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On the Non-Euclidean Geometry.

By William E. Story.

In volume IV of this Journal, pp. 332–335, I showed how the formulae of any Non-Euclidean plane trigonometry could be deduced from those of the Euclidean spherical trigonometry, namely by the replacement of each side by a certain constant multiple of that side and each angle by a certain constant multiple of that angle. In the present paper I propose to make the corresponding deduction for any Non-Euclidean spherical trigonometry, and incidentally to give a number of other formulæ relating to distances, angles, areas and volumes. Some of these formulæ exhibit an important principle which seems to me to be new, and which may be roughly expressed thus: *the distance* (*or angle*) *between any two geometrical elements* (points, planes or straight lines) *is, to a constant factor près, the same, in whatever way it is measured.* For example, the formulæ will show that the distance of a given point from the nearest point of a given plane is always proportional to the angle between the given plane and the nearest plane (*i. e.* that which makes the least angle with it) through the given point. Again, the least (or greatest) distance from a point of one of two given straight lines to a point of the other is proportional to the least (or greatest) angle which a plane through one of the straight lines makes with a plane through the other; and, if the lines intersect, this is again proportional to the angle between the lines. A similar theorem holds for the least (or greatest) distance between a point and a straight line and between a plane and a straight line, etc. Such greatest or least distances are of such frequent occurrence that it is convenient to speak of them without distinguishing between greatest or least; I therefore call such a distance a max.–min. distance, and later, designate a constant multiple of it, simply as the *distance* of two elements.

I give also expressions for the area of any spherical polygon, the circumference and area of any circle (the former was given by Gauss), the area of the surface and the volume of any sphere, and show that the double plane is identical with a sphere of quasi-infinite radius.

I assume the absolute given in the form

(1) $$\Omega \equiv x^2 + y^2 + z^2 - 1 = 0,$$

in point-coordinates, and

(2) $$\mho \equiv u^2 + v^2 + w^2 - 1 = 0$$

in tangential coordinates. I also write

$$\Omega_{ii} \equiv x_i^2 + y_i^2 + z_i^2 - 1, \quad \Omega_{ij} \equiv x_i x_j + y_i y_j + z_i z_j - 1,$$
$$\mho_{ii} \equiv u_i^2 + v_i^2 + w_i^2 - 1, \quad \mho_{ij} \equiv u_i u_j + v_i v_j + w_i w_j - 1.$$

As there are three flat elements, namely the point, the plane, and the straight line, there are six species of distance, each having its own unitary constant. Throughout this paper I shall use T, T' to denote points, M, M' planes, L, L' straight lines, and denote the non-Euclidean distance of two elements by the letters which represent them bound by a vinculum. It is rather convenient to designate angles as "distances." Then, for instance, the non-Euclidean distance (angle) between the plane M and the straight line L will be $\overline{ML}$.

In accordance with the principles laid down by Dr. Klein in his paper "Ueber die sogenannte Nicht-Euklidische Geometrie,"* I define the distance between two points to be k times the natural logarithm of the anharmonic ratio of the points with respect to the intersections of the absolute with their junction, and the distance between two planes as k' times their anharmonic ratio with respect to the tangent planes to the absolute through their intersection. According to these definitions, if

$$T = (x_1, y_1, z_1), \quad T' = (x_2, y_2, z_2), \quad M = (u_1, v_1, w_1), \quad M' = (u_2, v_2, w_2),$$

(3) $$\overline{TT'} = k \ln \left(\frac{\Omega_{12} + \sqrt{\Omega_{12}^2 - \Omega_{11}\Omega_{22}}}{\Omega_{12} - \sqrt{\Omega_{12}^2 - \Omega_{11}\Omega_{22}}} \right) = 2ik \cos^{-1} \left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}} \right),$$

(4) $$\overline{MM'} = k' \ln \left(\frac{\mho_{12} + \sqrt{\mho_{12}^2 - \mho_{11}\mho_{22}}}{\mho_{12} - \sqrt{\mho_{12}^2 - \mho_{11}\mho_{22}}} \right) = 2ik' \cos^{-1} \left(\frac{\mho_{12}}{\sqrt{\mho_{11}\mho_{22}}} \right),$$

the expressions as circular functions being those which I shall generally use.

The max.–min. distance of the point $T\ (x_1, y_1, z_1)$ from a point of the plane $M\ (u_1, v_1, w_1)$ is the max.–min. value of

$$2ik \cos^{-1} \left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}} \right)$$

where (x_2, y_2, z_2) satisfy the equation $u_1 x_2 + v_1 y_2 + w_1 z_2 + 1 = 0$; *i. e.* x_2, y_2, z_2 are determined by the conditions that

$$\Omega_{22}(x_1 dx_2 + y_1 dy_2 + z_1 dz_2) - \Omega_{12}(x_2 dx_2 + y_2 dy_2 + z_2 dz_2) = 0$$

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* Math. Annalen, Vol. IV, pp. 573–625.

shall be satisfied by every set of values of dx_2, dy_2, dz_2 which satisfy $u_1 dx_2 + v_1 dy_2 + w_1 dz_2 = 0$,
hence

$$\Omega_{22} x_1 - \Omega_{12} x_2 = \lambda u_1,$$
$$\Omega_{22} y_1 - \Omega_{12} y_2 = \lambda v_1,$$
$$\Omega_{22} z_1 - \Omega_{12} z_2 = \lambda w_1,$$

and

$$0 = \lambda(u_1 x_2 + v_1 y_2 + w_1 z_2 + 1) = \Omega_{22} - \Omega_{12} + \lambda,$$
$$\lambda = \Omega_{12} - \Omega_{22},$$
$$\lambda(u_1 x_1 + v_1 y_1 + w_1 z_1 + 1) = \Omega_{11}\Omega_{22} - \Omega_{12}^2 + \Omega_{22} - \Omega_{12} + \lambda,$$
$$\lambda \mho_{11} = \Omega_{22}(u_1 x_1 + v_1 y_1 + w_1 z_1 + 1) - \Omega_{22} + \Omega_{12} - \lambda,$$

i. e. writing $M_1 \equiv u_1 x_1 + v_1 y_1 + w_1 z_1 + 1$, we have

$$(\Omega_{12} - \Omega_{22}) M_1 = \Omega_{11}\Omega_{22} - \Omega_{12}^2,\ (\Omega_{12} - \Omega_{22}) \mho_{11} = \Omega_{22} M_1,$$

whence

$$\frac{\Omega_{12}}{\Omega_{22}} = \frac{\Omega_{11} + M_1}{\Omega_{12} + M_1} = \frac{M_1 + \mho_{11}}{\mho_{11}},\ \frac{\Omega_{12}}{\Omega_{11}} = \frac{\Omega_{11}\mho_{11} - M_1^2}{(\mho_{11} + M_1)\Omega_{11}},$$

and the max.–min. value required is

$$2ik \cos^{-1}\sqrt{\frac{\Omega_{11}\mho_{11} - M_1^2}{\Omega_{11}\mho_{11}}} = 2ik \sin^{-1}\left(\frac{\pm M_1}{\sqrt{\Omega_{11}\mho_{11}}}\right).$$

Similarly the max.–min. distance between the plane M and a plane through T is

$$2ik' \sin^{-1}\left(\frac{\pm M_1}{\sqrt{\Omega_{11}\mho_{11}}}\right),$$

as is also evident from the symmetry of this form. These distances being always proportional, I take

$$\overline{MT} = 2ik'' \sin^{-1}\left(\frac{M_1}{\sqrt{\Omega_{11}\mho_{11}}}\right). \tag{5}$$

On a nearer consideration of formulæ (3) and (4), it appears that the distance between two points and the max.–min. distance between two planes one through one point and one through the other are proportional, as are also the distance between two planes and the max.–min. distance between a point of the one and a point of the other.

If a point, plane, or straight line move continuously from a certain initial position to a certain final position, according to a definite law, the aggregate of the successive positions may be called a "course," and the measure of the total motion or "length of the course" may be defined to be the sum of the infinitesimal distances between successive positions of the point, plane, or straight line.

To obtain the length of any course we must express the distance between the points x, y, z and $x + dx, y + dy, z + dz$, between the planes u, v, w and $u + du, v + dv, w + dw$, or between the infinitely near positions of the straight line, and integrate between the two extremities of the course, taking account of the equation of the course. A course may be called "flat" or "doubly flat" according as its elements satisfy one or two linear equations. Thus a flat course from one point to another is a plane course, a doubly flat course from one point to another is a straight line, a flat course from one plane to another is an aggregate of positions of a plane always passing through a fixed point, and a doubly flat course from one plane to another is an aggregate of positions of a plane rotating about a fixed straight line in it. Then, as I prove below, the length of a course between two given points or planes is a max.–min. when the course is doubly flat.

The max.–min. distance of a given point $T(x_1, y_1, z_1)$ from a point of a given straight line L whose equation is

$$\frac{x - x_2}{A'} = \frac{y - y_2}{B'} = \frac{z - z_2}{C'}$$

is the distance of x_1, y_1, z_1 from the farthest or nearest point, say ξ, η, ζ of the line. Let

$$\xi = x_2 + \lambda A', \ \eta = y_2 + \lambda B', \ \zeta = z_2 + \lambda C',$$

then the condition for λ is

$$[\Omega_{22} + 2\lambda(A'x_2 + B'y_2 + C'z_2) + \lambda^2(A'^2 + B'^2 + C'^2)](A'x_1 + B'y_1 + C'z_1)d\lambda$$
$$- [\Omega_{12} + \lambda(A'x_1 + B'y_1 + C'z_1)][(A'x_2 + B'y_2 + C'z_2) + \lambda(A'^2 + B'^2 + C'^2)]d\lambda = 0$$

for all values of $d\lambda$, and the max.–min. distance is then

$$2ik \cos^{-1}\left(\frac{\Omega_{12} + \lambda(A'x_1 + B'y_1 + C'z_1)}{\sqrt{\Omega_{11}[\Omega_{22} + 2\lambda(A'x_2 + B'y_2 + C'z_2) + \lambda^2(A'^2 + B'^2 + C'^2)]}}\right)$$

$$= 2ik \cos^{-1}\sqrt{\frac{[\Omega_{12} + \lambda(A'x_1 + B'y_1 + C'z_1)](A'x_1 + B'y_1 + C'z_1)}{\Omega_{11}[(A'x_2 + B'y_2 + C'z_2) + \lambda(A'^2 + B'^2 + C'^2)]}}.$$

Writing

$$A'x_1 + B'y_1 + C'z_1 = N_1', \ A'x_2 + B'y_2 + C'z_2 = N_2', \ A'^2 + B'^2 + C'^2 = R'^2,$$

the condition for λ is

$$\lambda(R'^2\Omega_{12} - N_1'N_2') = N_1'\Omega_{22} - N_2'\Omega_{12},$$

and the max.–min. distance is

$$2ik \cos^{-1}\sqrt{\frac{R'^2\Omega_{12}^2 + N_1'^2\Omega_{22} - 2N_1'N_2'\Omega_{12}}{\Omega_{11}(R'^2\Omega_{22} - N_2'^2)}}.$$

The max.–min. distance of T from the farthest or nearest plane through L is 0, but the max.–min. distance of L from a straight line through T, as I shall show further on, is proportional to the value just obtained for the max.–min. distance of T from a point of L. I take therefore

$$(6)\ \overline{LT} = 2ik''' \cos^{-1}\sqrt{\frac{R'^2\Omega_{12}^2 + N_1'^2\Omega_{22} - 2N_1'N_2'\Omega_{12}}{\Omega_{11}(R'^2\Omega_{22} - N_2'^2)}} = 2ik''' \sin^{-1}\sqrt{\frac{\begin{vmatrix} \Omega_{11} & \Omega_{12} & N_1' \\ \Omega_{21} & \Omega_{22} & N_2' \\ N_1' & N_2' & R'^2 \end{vmatrix}}{\Omega_{11}(R'^2\Omega_{22} - N_2'^2)}}$$

$$= 2ik''' \cos^{-1}\sqrt{\frac{\Omega_{12}^2\Omega_{33} + \Omega_{13}^2\Omega_{22} - 2\Omega_{12}\Omega_{13}\Omega_{23}}{\Omega_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}} = 2ik''' \sin^{-1}\sqrt{\frac{\begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{vmatrix}}{\Omega_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}}$$

$$= 2ik''' \sin^{-1}\sqrt{\frac{N_T''^2\mho_{22} + R''^2T_2^2 - 2N_T''N_2''T_2}{\Omega_{11}(R''^2\mho_{22} - N_2''^2)}} = 2ik''' \cos^{-1}\sqrt{\frac{\begin{vmatrix} \Omega_{11} & T_2 & N_T'' \\ T_2 & \mho_{22} & N_2'' \\ N_T'' & N_2'' & R''^2 \end{vmatrix}}{\Omega_{11}(R''^2\mho_{22} - N_2''^2)}}$$

$$= 2ik''' \sin^{-1}\sqrt{\frac{T_2^2\mho_{33} + T_3^2\mho_{22} - 2T_2T_3\mho_{23}}{\Omega_{11}(\mho_{22}\mho_{33} - \mho_{23}^2)}} = 2ik''' \cos^{-1}\sqrt{\frac{\begin{vmatrix} \Omega_{11} & T_2 & T_3 \\ T_2 & \mho_{22} & \mho_{23} \\ T_3 & \mho_{23} & \mho_{33} \end{vmatrix}}{\Omega_{11}(\mho_{22}\mho_{33} - \mho_{23}^2)}}$$

In the third and fourth of these expressions L is supposed to be given as the junction of (x_2, y_2, z_2) and (x_3, y_3, z_3), in the fifth and sixth by equations of the form $\frac{u - u_2}{A''} = \frac{v - v_2}{B''} = \frac{w - w_2}{C''}$, and in the seventh and eighth as the intersection of (u_2, v_2, w_2) and (u_3, v_3, w_3); also
$N_T'' = A''x_1 + B''y_1 + C''z_1$, $T_2 = x_1u_2 + y_1v_2 + z_1w_2 + 1$, $R''^2 = A''^2 + B''^2 + C''^2$,
$N_2'' = A''u_2 + B''v_2 + C''w_2$.

The third and fourth forms are obtained from the first and second by putting $A' = x_3 - x_2$, $B' = y_3 - y_2$, $C' = z_3 - z_2$; the fifth and sixth are obtained from the first and second on eliminating A', B', C', x_2, y_2, z_2 by means of the equations

$$A' : B' : C' = B''w_2 - C''v_2 : C''u_2 - A''w_2 : A''v_2 - B''u_2,$$
$$u_2x_2 + v_2y_2 + w_2z_2 + 1 = 0, \quad A''x_2 + B''y_2 + C''z_2 = 0;$$

and the seventh and eighth are obtained from the fifth and sixth by putting

$$A'' : B'' : C'' = u_3 - u_2 : v_3 - v_2 : w_3 - w_2.$$

If (u_2, v_2, w_2) is the plane of T and L, $T_2 = 0$, and

$$(7)\quad \overline{LT} = 2ik''' \sin^{-1}\left(\frac{\pm N_T''\sqrt{\mho_{22}}}{\sqrt{\Omega_{11}(R''^2\mho_{22} - N_2''^2)}}\right) = 2ik''' \sin^{-1}\left(\frac{\pm T_3\sqrt{\mho_{22}}}{\sqrt{\Omega_{11}(\mho_{22}\mho_{33} - \mho_{23}^2)}}\right).$$

The max.–min. distance between the plane $M(u_1, v_1, w_1)$ and a plane through the straight line L whose equation is

$$\frac{u-u_2}{A''}=\frac{v-v_2}{B''}=\frac{w-w_2}{C''}$$

is $$2ik' \cos^{-1}\sqrt{\frac{R''^2 \mho_{12}^2 + N_1''^2 \mho_{22} - 2N_1'' N_2'' \mho_{12}}{\mho_{11}(R''^2 \mho_{22} - N_2''^2)}},$$

where

$$N_1'' = A''u_1 + B''v_1 + C''w_1, \quad N_2'' = A''u_2 + B''v_2 + C''w_2, \quad R''^2 = A''^2 + B''^2 + C''^2.$$

If the equation of L is given in the form

$$\frac{x-x_2}{A'}=\frac{y-y_2}{B'}=\frac{z-z_2}{C'},$$

then $$A'' : B'' : C'' = B'z_2 - C'y_2 : C'x_2 - A'z_2 : A'y_2 - B'x_2;$$

and putting $$A'u_1 + B'v_1 + C'w_1 = N'_M, \quad u_1x_2 + v_1y_2 + w_1z_2 + 1 = M_2,$$

the max.–min. distance will be found to be

$$2ik' \sin^{-1}\sqrt{\frac{N_M'^2 \Omega_{22} + R'^2 M_2^2 - 2N'_M M_2 N_2'}{\mho_{11}(R'^2 \Omega_{22} - N_2'^2)}}.$$

If the straight line L were given as the intersection of the planes (u_2, v_2, w_2) and (u_3, v_3, w_3), the max.–min. distance between the plane M and a plane through L is the max.–min. value of

$$2ik' \cos^{-1}\left(\frac{\mho_{12} + \lambda\mho_{13}}{\sqrt{\mho_{11}(\mho_{22} + 2\lambda\mho_{23} + \lambda^2\mho_{33})}}\right);$$

the condition for λ is

$$\lambda(\mho_{12}\mho_{33} - \mho_{13}\mho_{23}) + (\mho_{12}\mho_{23} - \mho_{13}\mho_{22}) = 0,$$

and the max.–min. distance required is

$$2ik' \cos^{-1}\sqrt{\frac{\mho_{12}^2\mho_{33} + \mho_{13}^2\mho_{22} - 2\mho_{12}\mho_{13}\mho_{23}}{\mho_{11}(\mho_{22}\mho_{33} - \mho_{23}^2)}}.$$

The max.–min. distance between M and a point of L is 0, as is also the max.–min. distance between L and a point of M. As I shall afterwards show the max.–min. distance between L and a line in M is proportional to the above given max.–min. distance between M and a plane through L. I take, therefore,

$$(8) \quad \overline{ML} = 2ik^{\text{IV}} \cos^{-1}\sqrt{\frac{\mho_{12}^2\mho_{33} + \mho_{13}^2\mho_{22} - 2\mho_{12}\mho_{13}\mho_{23}}{\mho_{11}(\mho_{22}\mho_{33} - \mho_{23}^2)}} = 2ik^{\text{IV}} \sin^{-1}\sqrt{\frac{\begin{vmatrix} \mho_{11} & \mho_{12} & \mho_{13} \\ \mho_{21} & \mho_{22} & \mho_{23} \\ \mho_{31} & \mho_{32} & \mho_{33} \end{vmatrix}}{\mho_{11}(\mho_{22}\mho_{33} - \mho_{23}^2)}}$$

$$= 2ik^{\text{IV}} \cos^{-1}\sqrt{\frac{R''^2\mho_{12}^2 + N_1''^2\mho_{22} - 2N_1''N_2''\mho_{12}}{\mho_{11}(R''^2\mho_{22} - N_2''^2)}} = 2ik^{\text{IV}} \sin^{-1}\sqrt{\frac{\begin{vmatrix} \mho_{11} & \mho_{12} & N_1'' \\ \mho_{21} & \mho_{22} & N_2'' \\ N_1'' & N_2'' & R''^2 \end{vmatrix}}{\mho_{11}(R''^2\mho_{22} - N_2''^2)}}$$

$$= 2ik^{\mathrm{IV}} \sin^{-1}\sqrt{\frac{N_M'^2\Omega_{22} + R'^2M_2^2 - 2N_M'N_2'M_2}{\mho_{11}(R'^2\Omega_{22} - N_2'^2)}} = 2ik^{\mathrm{IV}} \cos^{-1}\sqrt{\frac{\begin{vmatrix} \mho_{11} & M_2 & N_M' \\ M_2 & \Omega_{22} & N_2' \\ N_M' & N_2' & R'^2 \end{vmatrix}}{\mho_{11}(R'^2\Omega_{22} - N_2'^2)}}$$

$$= 2ik^{\mathrm{IV}} \sin^{-1}\sqrt{\frac{M_2^2\Omega_{33} - 2M_2M_3\Omega_{23} + M_3^2\Omega_{22}}{\mho_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}} = 2ik^{\mathrm{IV}} \cos^{-1}\sqrt{\frac{\begin{vmatrix} \mho_{11} & M_2 & M_3 \\ M_2 & \Omega_{22} & \Omega_{23} \\ M_3 & \Omega_{32} & \Omega_{33} \end{vmatrix}}{\mho_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}}.$$

In the fifth and sixth of these expressions L is supposed to be given in the form $\frac{x - x_2}{A'} = \frac{y - y_2}{B'} = \frac{z - z_2}{C'}$, and in the seventh and eighth expressions L is determined as the junction of (x_2, y_2, z_2) and (x_3, y_3, z_3).

The fifth and sixth are obtained from the first and second by eliminating $u_2, v_2, w_2, u_3, v_3, w_3$ by means of the equations

$$u_2x_2 + v_2y_2 + w_2z_2 + 1 = 0, \qquad u_3x_2 + v_3y_2 + w_3z_2 + 1 = 0,$$
$$A'u_2 + B'v_2 + C'w_2 = 0, \qquad A'u_3 + B'v_3 + C'w_3 = 0;$$

inasmuch as (u_2, v_2, w_2) and (u_3, v_3, w_3) are any planes through L, it is allowable to assume $v_2 = 0$, $u_3 = 0$, which simplifies the reduction. The seventh and eighth forms of (8) are obtained from the fifth and sixth by writing $A' = x_3 - x_2$, $B' = y_3 - y_2$, $C' = z_3 - z_2$.

If x_2, y_2, z_2 is the point in which the plane M is met by the line L, $M_2 = 0$, and

$$\text{(9)} \qquad \overline{ML} = 2ik^{\mathrm{IV}} \sin^{-1}\left(\frac{\pm M_3\sqrt{\Omega_{22}}}{\sqrt{\mho_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2)}}\right)$$
$$= 2ik^{\mathrm{IV}} \sin^{-1}\left(\frac{\pm N_M'\sqrt{\Omega_{22}}}{\sqrt{\mho_{11}(R'^2\Omega_{22} - N_2'^2)}}\right).$$

The max.–min. distance between the line L and a point of the line L', where L is the junction of (x_1, y_1, z_1) and (x_2, y_2, z_2), and L' the junction of (x_3, y_3, z_3) and (x_4, y_4, z_4) is the max.–min. distance between L and the point

$$\left(\frac{x_3 + \lambda x_4}{1 + \lambda}, \frac{y_3 + \lambda y_4}{1 + \lambda}, \frac{z_3 + \lambda z_4}{1 + \lambda}\right)$$

i. e. by (6) the max.–min. value of

$$2ik''' \cos^{-1}\sqrt{\frac{P + 2\lambda Q + \lambda^2 R}{A(\Omega_{33} + 2\lambda\Omega_{34} + \lambda^2\Omega_{44})}},$$

where

$$P = \Omega_{13}^2\Omega_{22} + \Omega_{23}^2\Omega_{11} - 2\Omega_{13}\Omega_{23}\Omega_{12},$$
$$Q = \Omega_{13}\Omega_{14}\Omega_{22} + \Omega_{23}\Omega_{24}\Omega_{11} - \Omega_{13}\Omega_{24}\Omega_{12} - \Omega_{14}\Omega_{23}\Omega_{12},$$
$$R = \Omega_{14}^2\Omega_{22} + \Omega_{24}^2\Omega_{11} - 2\Omega_{14}\Omega_{24}\Omega_{12},$$
$$A = \Omega_{11}\Omega_{22} - \Omega_{12}^2,$$

I also put

$$B = \Omega_{13}\Omega_{24} - \Omega_{14}\Omega_{23}, \quad C = \Omega_{33}\Omega_{44} - \Omega_{34}^2,$$
$$\alpha = \Omega_{11}\Omega_{23} - \Omega_{12}\Omega_{13}, \quad \beta = \Omega_{11}\Omega_{24} - \Omega_{12}\Omega_{14},$$
$$\gamma = \Omega_{12}\Omega_{23} - \Omega_{13}\Omega_{22}, \quad \delta = \Omega_{12}\Omega_{24} - \Omega_{14}\Omega_{22},$$
$$\alpha' = \Omega_{23}\Omega_{44} - \Omega_{24}\Omega_{34}, \quad \beta' = \Omega_{24}\Omega_{33} - \Omega_{23}\Omega_{34},$$
$$\gamma' = \Omega_{14}\Omega_{34} - \Omega_{13}\Omega_{44}, \quad \delta' = \Omega_{13}\Omega_{34} - \Omega_{14}\Omega_{33};$$

namely A, B, α, β, γ, δ are the minors of the second order formed from the first and second rows of the determinant

$$\begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} \end{vmatrix},$$

and B, C, α', β', γ', δ' are the minors formed from the third and fourth rows, *i. e.* this determinant, expressed as a sum of products of minors formed from the first two and last two rows, is

$$AC + B^2 - \alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta';$$

but the determinant is evidently $-\Delta^2$, where

$$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

(*i. e.* $\Delta = 0$ is the condition that L and L' intersect, or lie in one plane), hence

$$(10) \qquad -\Delta^2 = AC + B^2 - \alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'.$$

Now the condition for λ is

$$(\Omega_{33} + 2\lambda\Omega_{34} + \lambda^2\Omega_{44})(Q + \lambda R) - (P + 2\lambda Q + \lambda^2 R)(\Omega_{34} + \lambda\Omega_{44}) = 0,$$

and from this follows

$$\frac{P + 2\lambda Q + \lambda^2 R}{\Omega_{33} + 2\lambda\Omega_{34} + \lambda^2\Omega_{44}} = \frac{Q + \lambda R}{\Omega_{34} + \lambda\Omega_{44}},$$

and hence the max.–min. value required is

$$2ik'' \cos^{-1}\sqrt{\frac{Q + \lambda R}{A(\Omega_{34} + \lambda\Omega_{44})}}.$$

Put

$$t = \frac{Q + \lambda R}{\Omega_{34} + \lambda\Omega_{44}} = \frac{P + \lambda Q}{\Omega_{33} + \lambda\Omega_{34}},$$

by the previous equation; hence

$$\lambda = -\frac{t\Omega_{34} - Q}{t\Omega_{44} - R} = -\frac{t\Omega_{33} - P}{t\Omega_{34} - Q},$$

i. e. t is determined by the quadratic

$$Ct^2-(\Omega_{33}R+\Omega_{44}P-2\Omega_{34}Q)t+PR-Q^2=0;$$

but

$$\Omega_{33}R-\Omega_{34}Q=\beta\beta'+\delta\delta',$$
$$\Omega_{44}P-\Omega_{34}Q=\alpha\alpha'+\gamma\gamma',$$

hence

$$\Omega_{33}R+\Omega_{44}P-2\Omega_{34}Q=\alpha\alpha'+\beta\beta'+\gamma\gamma'+\delta\delta'=AC+B^2+\Delta^2,$$

by (10); also P, Q, R may be written

$$P=\Omega_{23}\alpha-\Omega_{13}\gamma,\ R=\Omega_{24}\beta-\Omega_{14}\delta,$$
$$Q=\Omega_{23}\beta-\Omega_{13}\delta=\Omega_{24}\alpha-\Omega_{14}\gamma,$$

hence

$$PR-Q^2=\begin{vmatrix}\Omega_{23}\alpha-\Omega_{13}\gamma, & \Omega_{24}\alpha-\Omega_{14}\gamma\\ \Omega_{23}\beta-\Omega_{13}\delta, & \Omega_{24}\beta-\Omega_{14}\delta\end{vmatrix}=B\begin{vmatrix}\alpha & \gamma\\ \beta & \delta\end{vmatrix}=AB^2;$$

the quadratic in t then becomes

$$Ct^2-(AC+B^2+\Delta^2)t+AB^2=0,$$

i. e. the max.–min. distance required is

$$2ik'''\cos^{-1}\sqrt{\frac{t}{A}}=2ik'''\cos^{-1}\sqrt{\frac{AC+B^2+\Delta^2\pm\sqrt{(AC+B^2+\Delta^2)^2-4AB^2C}}{2AC}}$$
$$=2ik'''\cos^{-1}\sqrt{\frac{AC+B^2+\Delta^2\pm\sqrt{(AC-B^2+\Delta^2)+4B^2\Delta^2}}{2AC}}$$
$$=2ik'''\sin^{-1}\sqrt{\frac{AC-B^2-\Delta^2\mp\sqrt{(AC-B^2-\Delta^2)^2+4AC\Delta^2}}{2AC}};$$

the symmetry of this expression shows that it is also the max.–min. distance between L' and a point of L.

It is very evident, from the investigation leading to (6), that the max.–min. distance between a point of L and a point of L' will be found from this max.–min. distance between L and a point of L' by changing k''' to k, *i. e.* this distance is

$$2ik\sin^{-1}\sqrt{\frac{AC-B^2-\Delta^2\mp\sqrt{(AC-B^2-\Delta^2)^2+4AC\Delta^2}}{2AC}}.$$

Similarly, if L is the intersection of the planes (u_1, v_1, w_1) and (u_2, v_2, w_2) and L' the intersection of (u_3, v_3, w_3) and (u_4, v_4, w_4), the max.–min. distance between L and a plane through L', or between L' and a plane through L is

$$2ik^{\mathrm{IV}}\sin^{-1}\sqrt{\frac{\rotatebox{180}{A}\rotatebox{180}{C}-\rotatebox{180}{B}^2-\rotatebox{180}{Δ}^2\mp\sqrt{(\rotatebox{180}{A}\rotatebox{180}{C}-\rotatebox{180}{B}^2-\rotatebox{180}{Δ}^2)^2+4\rotatebox{180}{A}\rotatebox{180}{C}\rotatebox{180}{Δ}^2}}{2\rotatebox{180}{A}\rotatebox{180}{C}}},$$

where

$$\rotatebox{180}{A} = \mho_{11}\mho_{22} - \mho_{12}^2, \quad \rotatebox{180}{B} = \mho_{13}\mho_{24} - \mho_{14}\mho_{23}, \quad \rotatebox{180}{C} = \mho_{33}\mho_{44} - \mho_{34}^2,$$

$$\nabla = \begin{vmatrix} u_1 & v_1 & w_1 & -1 \\ u_2 & v_2 & w_2 & -1 \\ u_3 & v_3 & w_3 & -1 \\ u_4 & v_4 & w_4 & -1 \end{vmatrix}.$$

The max.–min. distance between a plane through L and a plane through L' is then

$$2ik' \sin^{-1}\sqrt{\frac{\rotatebox{180}{A}\rotatebox{180}{C} - \rotatebox{180}{B}^2 - \nabla^2 \mp \sqrt{(\rotatebox{180}{A}\rotatebox{180}{C} - \rotatebox{180}{B}^2 - \nabla^2)^2 + 4\rotatebox{180}{A}\rotatebox{180}{C}\nabla}}{2\rotatebox{180}{A}\rotatebox{180}{C}}}.$$

The max.–min. distance between a point on L_1, the junction of (x_1, y_1, z_1) and (x_2, y_2, z_2), and a plane through L_2, the intersection of (u_3, v_3, w_3) and (u_4, v_4, w_4), is the max.–min. value of

$$2ik'' \sin^{-1} \frac{M_1''' + \lambda M_2''' + \mu M_1^{\text{IV}} + \lambda\mu M_2^{\text{IV}}}{\sqrt{(\Omega_{11} + 2\lambda\Omega_{12} + \lambda^2\Omega_{22})(\mho_{33} + 2\mu\mho_{34} + \mu^2\mho_{44})}},$$

where $M_i''' = u_3x_i + v_3y_i + w_3z_i + 1$, $M_i^{\text{IV}} = u_4x_i + v_4y_i + w_4z_i + 1$,
but this expression is obtained from

$$2ik \cos^{-1} \frac{\Omega_{13} + \lambda\Omega_{23} + \mu\Omega_{14} + \lambda\mu\Omega_{24}}{\sqrt{(\Omega_{11} + 2\lambda\Omega_{12} + \lambda^2\Omega_{22})(\Omega_{33} + 2\mu\Omega_{34} + \mu^2\Omega_{44})}}$$

by replacing Ω_{13}, Ω_{23}, Ω_{14}, Ω_{24}, Ω_{33}, Ω_{34}, Ω_{44}, k, $\cos^{-1}$ by M_1''', M_2''', M_1^{IV}, M_2^{IV}, $\mho_{33}$, $\mho_{34}$, $\mho_{44}$, k'', $\sin^{-1}$; now the max.–min. value of the latter expression is the max.–min. distance between a point of the junction of (x_1, y_1, z_1) and (x_2, y_2, z_2) and a point of the junction of (x_3, y_3, z_3) and (x_4, y_4, z_4); but this distance has been found to be

$$2ik \sin^{-1}\sqrt{\frac{AC - B^2 - \Delta^2 \mp \sqrt{(AC - B^2 - \Delta^2)^2 + 4AC\Delta^2}}{2AC}}$$

where A, C, B, Δ are defined as above; hence the max.–min. distance between a point on L and a plane through L' is

$$2ik'' \cos^{-1}\sqrt{\frac{A\rotatebox{180}{C} - \mathfrak{B}^2 - \mathfrak{D}^2 \mp \sqrt{(A\rotatebox{180}{C} - \mathfrak{B}^2 - \mathfrak{D}^2)^2 + 4A\rotatebox{180}{C}\mathfrak{D}^2}}{2A\rotatebox{180}{C}}}$$

$$= 2ik'' \sin^{-1}\sqrt{\frac{A\rotatebox{180}{C} + \mathfrak{B}^2 + \mathfrak{D}^2 \pm \sqrt{(A\rotatebox{180}{C} + \mathfrak{B}^2 + \mathfrak{D}^2)^2 - 4A\rotatebox{180}{C}\mathfrak{B}^2}}{2A\rotatebox{180}{C}}},$$

where A, $\rotatebox{180}{$C$}$ are defined as above, and

$$\mathfrak{B} = M_1'''M_2^{\text{IV}} - M_1^{\text{IV}}M_2''',$$

$$-\mathfrak{D}^2 = \begin{vmatrix} \Omega_{11} & \Omega_{12} & M_1''' & M_1^{\text{IV}} \\ \Omega_{21} & \Omega_{22} & M_2''' & M_2^{\text{IV}} \\ M_1''' & M_2''' & \Omega_{33} & \Omega_{34} \\ M_1^{\text{IV}} & M_2^{\text{IV}} & \Omega_{43} & \Omega_{44} \end{vmatrix} = - \begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ u_3, & v_3, & w_3, & -1 \\ u_4, & v_4, & w_4, & -1 \end{vmatrix}^2,$$

i. e.

$$\mathfrak{D} = \begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ u_3, & v_3, & w_3, & -1 \\ u_4, & v_4, & w_4, & -1 \end{vmatrix},$$

and $\mathfrak{D} = 0$ is the condition that each of the lines L and L' should intersect the polar line of the other with respect to the absolute, *i. e.* the condition for what I shall afterwards call the perpendicularity of the lines. The proportionality of the last three distances can be best proven by expressing them in terms of the coordinates of the lines; λ, μ, ρ, σ are simply proportional factors, let the coordinates of L and L' be respectively a_1, b_1, c_1, f_1, g_1, h_1 and a_2, b_2, c_2, f_2, g_2, h_2; then

$$\begin{aligned} &\lambda(y_1z_2 - z_1y_2) = \mu(u_1 - u_2) = a_1, \ \lambda(z_1x_2 - x_1z_2) = \mu(v_1 - v_2) = b_1, \\ &\lambda(x_1y_2 - y_1x_2) = \mu(w_1 - w_2) = c_1, \ \lambda(x_1 - x_2) = \mu(v_1w_2 - w_1v_2) = f_1, \\ &\lambda(y_1 - y_2) = \mu(w_1u_2 - u_1w_2) = g_1, \ \lambda(z_1 - z_2) = \mu(u_1v_2 - v_1u_2) = h_1, \\ &\rho(y_3z_4 - z_3y_4) = \sigma(u_3 - u_4) = a_2, \ \rho(z_3x_4 - x_3z_4) = \sigma(v_3 - v_4) = b_2, \\ &\rho(x_3y_4 - y_3x_4) = \sigma(w_3 - w_4) = c_2, \ \rho(x_3 - x_4) = \sigma(v_3w_4 - w_3v_4) = f_2, \\ &\rho(y_3 - y_4) = \sigma(w_3u_4 - u_3w_4) = g_2, \ \rho(z_3 - z_4) = \sigma(u_3v_4 - v_3u_4) = h_2; \end{aligned}$$

and

$$\begin{aligned} &\lambda^2A = a_1^2 + b_1^2 + c_1^2 - f_1^2 - g_1^2 - h_1^2, \ \lambda\rho B = a_1a_2 + b_1b_2 + c_1c_2 - f_1f_2 - g_1g_2 - h_1h_2, \\ &\rho^2C = a_2^2 + b_2^2 + c_2^2 - f_2^2 - g_2^2 - h_2^2, \ \lambda\rho\Delta = a_1f_2 + b_1g_2 + c_1h_2 + f_1a_2 + g_1b_2 + h_1c_2, \\ &\mu^2\forall = f_1^2 + g_1^2 + h_1^2 - a_1^2 - b_1^2 - c_1^2, \ \mu\sigma\rotatebox{180}{B} = f_1f_2 + g_1g_2 + h_1h_2 - a_1a_2 - b_1b_2 - c_1c_2, \\ &\sigma^2\rotatebox{180}{C} = f_2^2 + g_2^2 + h_2^2 - a_2^2 - b_2^2 - c_2^2, \ \mu\sigma\nabla = -a_1f_2 - b_1g_2 - c_1h_2 - f_1a_2 - g_1b_2 - h_1c_2, \\ &\lambda\sigma\mathfrak{B} = a_1f_2 + b_1g_2 + c_1h_2 + f_1a_2 + g_1b_2 + h_1c_2, \ \lambda\sigma\mathfrak{D} = f_1f_2 + g_1g_2 + h_1h_2 - a_1a_2 - b_1b_2 - c_1c_2, \end{aligned}$$

from which follow

$$\mu^2\forall = -\lambda^2A, \ \mu\sigma\rotatebox{180}{B} = \lambda\sigma\mathfrak{D} = -\lambda\rho B, \ \sigma^2\rotatebox{180}{C} = -\rho^2C, \ \mu\sigma\nabla = -\lambda\sigma\mathfrak{B} = -\lambda\rho\Delta;$$

hence

$$\begin{aligned} &\mu^2\sigma^2(\forall\rotatebox{180}{C} - \rotatebox{180}{B}^2 - \nabla^2) = \lambda^2\rho^2(AC - B^2 - \Delta^2), \\ &\mu^4\sigma^4\forall\rotatebox{180}{C}\nabla^2 = \lambda^4\rho^4AC\Delta^2, \ \mu^2\sigma^2\forall\rotatebox{180}{C} = \lambda^2\rho^2AC, \\ &\lambda^2\sigma^2(A\rotatebox{180}{C} + \mathfrak{B}^2 + \mathfrak{D}^2) = -\lambda^2\rho^2(AC - B^2 - \Delta^2), \\ &\lambda^4\sigma^4A\rotatebox{180}{C}\mathfrak{D}^2 = -\lambda^4\rho^4ACB^2, \ \lambda^2\sigma^2A\rotatebox{180}{C} = -\lambda^2\rho^2AC, \end{aligned}$$

and therefore

$$\frac{\rotatebox{180}{A}\rotatebox{180}{C}-\rotatebox{180}{B}^2-\nabla^2\mp\sqrt{(\rotatebox{180}{A}\rotatebox{180}{C}-\rotatebox{180}{B}^2-\nabla^2)^2+4\rotatebox{180}{A}\rotatebox{180}{C}\nabla^2}}{2\rotatebox{180}{A}\rotatebox{180}{C}}$$

$$=-\frac{A\rotatebox{180}{C}+\mathfrak{B}^2+\mathfrak{D}^2\pm\sqrt{(A\rotatebox{180}{C}+\mathfrak{B}^2+\mathfrak{D}^2)^2-4A\rotatebox{180}{C}\mathfrak{B}^2}}{2A\rotatebox{180}{C}}$$

$$=\frac{AC-B^2-\Delta^2\mp\sqrt{(AC-B^2-\Delta^2)^2+4AC\Delta^2}}{2AC},$$

and the proportionality of the three distances is established. I therefore take

$$(11)\qquad \overline{LL'}=2ik^{\text{v}}\sin^{-1}\sqrt{\frac{AC-B^2-\Delta^2\mp\sqrt{(AC-B^2-\Delta^2)^2+4AC\Delta^2}}{2AC}},$$

in which the quantity under the outer radical sign may be replaced by either of the above given equal expressions. It is to be noticed that, on account of the double sign before the inner radical, there are two max.–min. distances between two given straight lines, corresponding, as I shall show below, to the geometrical fact that two straight lines can be drawn to intersect four given straight lines.

If the straight lines L and L' intersect, Δ, ∇ and $\mathfrak{B}$ vanish, one of the values of $\overline{LL'}$ is 0, and the other is

$$(12)\qquad \overline{LL'}=2ik^{\text{v}}\sin^{-1}\sqrt{\frac{AC-B^2}{AC}}=2ik^{\text{v}}\cos^{-1}\frac{B}{\sqrt{AC}}=2ik^{\text{v}}\cos^{-1}\frac{-\rotatebox{180}{B}}{\sqrt{\rotatebox{180}{A}\rotatebox{180}{C}}}$$

$$=2ik^{\text{v}}\cos^{-1}\frac{-\mathfrak{D}}{\sqrt{-A\rotatebox{180}{C}}};$$

now

$$2i\cos^{-1}\frac{B}{\sqrt{AC}}=\ln\frac{B\pm\sqrt{B^2-AC}}{B\mp\sqrt{B^2-AC}}=\ln\alpha,\text{ say,}$$

where α is easily seen to be the anharmonic ratio of L and L' referred to the two tangents to the absolute from their intersection, which lie in their plane; namely, the above angle is evidently independent of the particular choice of the points T_1, T_2, T_3, T_4 in either line, hence the value of α will be unaffected by the assumption that T_2 and T_4 coincide with the intersection of the lines; now α is the ratio of the roots of the equation

$$A+2\lambda B+\lambda^2C=0,$$

and if $T_4=T_2$ this is

$$(\Omega_{11}\Omega_{22}-\Omega_{12}^2)+2\lambda(\Omega_{13}\Omega_{22}-\Omega_{12}\Omega_{23})+\lambda^2(\Omega_{33}\Omega_{22}-\Omega_{23}^2)=0,$$

i. e.

$$\Omega_{22}(\Omega_{11}+2\lambda\Omega_{13}+\lambda^2\Omega_{33})-(\Omega_{12}+\lambda\Omega_{23})^2=0,$$

but this is the condition that the point $(x_1+\lambda x_3,\ y_1+\lambda y_3,\ z_1+\lambda z_3)$ shall lie on a tangent from T_2 to the absolute, *i. e.* α is the anharmonic ratio of T_1, T_3 with respect to the intersections of the line T_1T_3 with the tangent cone from T_2 to the absolute, but this is the anharmonic ratio of the lines L and L', to which reference is made above.

We can now find the max.–min. distances, above mentioned, between a given straight line and the nearest or farthest line through a given point, and between a given straight line and the nearest or farthest line in a given plane. It suffices to determine the max.–min. value of (11) when L and (x_3, y_3, z_3) are constant and (x_4, y_4, z_4) variable, and when L and (u_3, v_3, w_3) are constant and (u_4, v_4, w_4) variable. In the former case, evidently

$$\frac{B^2+\varDelta^2}{AC} \mp \sqrt{\left(1-\frac{B^2+\varDelta^2}{AC}\right)^2+\frac{4\varDelta^2}{AC}}$$

is to be a max.–min.; but there are two independent parameters involved in the determination of a straight line through a given point, and taking $\frac{B^2+\varDelta^2}{AC}$ and $\frac{\varDelta^2}{AC}$ to be the two parameters it appears that $\Delta = 0$ or $AC = 0$, but the latter condition makes the max.–min. distance indeterminate, and assuming $\Delta = 0$ there is still a parameter to be determined, which may be done thus: because the lines intersect we may put $x_4 : y_4 : z_4 : 1 = x_1 + \lambda x_2 : y_1 + \lambda y_2 : z_1 + \lambda z_2 : 1 + \lambda$, and the expression whose max.–min. value is to be determined is $2ik^{\text{v}} \cos^{-1} \frac{B}{\sqrt{AC}}$, for which we easily find $\lambda = -\frac{\varOmega_{13}}{\varOmega_{23}}$, and the max.–min. value is

$$2ik^{\text{v}} \cos \sqrt{\frac{\varOmega_{11}\varOmega_{23}^2 + \varOmega_{22}\varOmega_{13}^2 - 2\varOmega_{12}\varOmega_{13}\varOmega_{23}}{\varOmega_{33}(\varOmega_{11}\varOmega_{22} - \varOmega_{12}^2)}},$$

which is proportional to the value which would be obtained from (6) for the distance between the line and the point, namely by an interchange of the suffices 1 and 3. In the same manner the proportionality of the max.–min. distance between a given line and a line in a given plane to the expression obtained from (8) for the distance between the line and the plane.

All the above given formulæ for distance may easily be applied to the determination of the distance of two infinitely near elements, but the most interesting case of this kind, from the point of view of the Euclidean Geometry, is that in which the elements are points, say $T(x, y, z)$ and $T'(x+dx, y+dy, z+dz)$. For convenience I write

$$\delta\Omega = \frac{1}{2}\left(\frac{\partial \varOmega}{\partial x}dx + \frac{\partial \varOmega}{\partial y}dy + \frac{\partial \varOmega}{\partial z}dz\right),$$

$$\delta^2\Omega = \frac{1}{2}\left(\frac{\partial^2 \varOmega}{\partial x^2}dx^2 + \frac{\partial^2 \varOmega}{\partial y^2}dy^2 + \frac{\partial^2 \varOmega}{\partial z^2}dz^2 + 2\frac{\partial^2 \varOmega}{\partial y \partial z}dydz + 2\frac{\partial^2 \varOmega}{\partial z \partial x}dzdx + 2\frac{\partial^2 \varOmega}{\partial x \partial y}dxdy\right);$$

then the distance is

$$(13)\qquad d\overline{TT'} = 2ik\cos^{-1}\frac{\Omega+\delta\Omega}{\sqrt{\Omega(\Omega+2\delta\Omega+\delta^2\Omega)}}$$

$$= 2ik\sin^{-1}\sqrt{\frac{\Omega\delta^2\Omega-(\delta\Omega)^2}{\Omega(\Omega+2\delta\Omega+\delta^2\Omega)}}$$

$$= 2ik\frac{1}{\Omega}\sqrt{\Omega\delta^2\Omega-(\delta\Omega)^2},$$

neglecting higher powers of dx, dy, dz. This may be put into the short symbolic form

$$d\overline{TT'} = 2ik\sqrt{\delta\left(\frac{\delta\Omega}{\Omega}\right)} = 2ik\sqrt{\delta^2\ln\Omega}.$$

Similarly the distance between the planes $M(u, v, w)$ and $M'(u+du, v+dv, w+dw)$, where du, dv, dw are infinitesimal, is

$$(14)\qquad d\overline{MM'} = 2ik'\frac{1}{\mho}\sqrt{\mho\delta^2\mho-(\delta\mho)^2} = 2ik'\sqrt{\delta^2\ln\mho},$$

where

$$\delta\mho = \frac{1}{2}\left(\frac{\partial\mho}{\partial u}du+\frac{\partial\mho}{\partial v}dv+\frac{\partial\mho}{\partial w}dw\right)$$

$$\delta^2\mho = \frac{1}{2}\left(\frac{\partial^2\mho}{\partial u^2}du^2+\frac{\partial^2\mho}{\partial v^2}dv^2+\frac{\partial^2\mho}{\partial w^2}dw^2+2\frac{\partial^2\mho}{\partial v\partial w}dv\,dw\right.$$
$$\left.+2\frac{\partial^2\mho}{\partial w\partial u}dw\,du+2\frac{\partial^2\mho}{\partial u\partial v}du\,dv\right).$$

The distance between two infinitely near straight lines L and L', given by their coordinates (a, b, c, f, g, h) and $(a+da, b+db, c+dc, f+df, g+dg, h+dh)$ is obtained from (11) by substituting for A, B, C, Δ their values in terms of line coordinates. Then

$$B = A+\delta A,$$
$$C = A+2\delta A+\delta^2 A,$$
$$\Delta = 2\delta O,$$

where

$$O \equiv af+bg+ch = 0,$$
$$\delta \equiv \frac{1}{2}\left(\frac{\partial}{\partial a}da+\frac{\partial}{\partial b}db+\frac{\partial}{\partial c}dc+\frac{\partial}{\partial f}df+\frac{\partial}{\partial g}dg+\frac{\partial}{\partial h}dh\right),$$
$$\delta^2 \equiv \frac{1}{2}\left(\frac{\partial^2}{\partial a^2}da^2+\ldots+2\frac{\partial^2}{\partial a\partial f}da\,df+\ldots\right).$$

Now $O=0$ for any line, and $2\delta O = -\delta^2 O$ is a differential of the second order; and it will be found that

$$(15)\qquad d\overline{LL'} = ik^{\mathrm{v}}\left(\frac{A\delta^2A-(\delta A)^2\pm\sqrt{[A\delta^2A-(\delta A)^2]^2+4A^2(\delta^2O)^2}}{A^2}\right),$$

when terms of a higher than the second order are neglected.

The distance between any two infinitely near positions in a "course" will be given by (13), (14) or (15), and the "*length of the course*" will be obtained by integrating this expression between the extreme positions. It will be found by the usual method of the calculus of variations that the shortest course between any two given points or planes is "*doubly flat*"; the integrals in these cases are (3) and (4) respectively.

The same thing may be proved thus: the condition that the sum of the distances of a variable point (x_3, y_3, z_3) from two fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) as determined by (3) shall be a maximum or minimum is given by three equations of the fourth degree in (x_3, y_3, z_3), but they may be written as linear homogeneous equations in

$$\left(\frac{\partial\Omega_{11}}{\partial x_1}, \frac{\partial\Omega_{22}}{\partial x_2}, \frac{\partial\Omega_{33}}{\partial x_3}\right), \left(\frac{\partial\Omega_{11}}{\partial y_1}, \frac{\partial\Omega_{22}}{\partial y_2}, \frac{\partial\Omega_{33}}{\partial y_3}\right) \text{ and } \left(\frac{\partial\Omega_{11}}{\partial z_1}, \frac{\partial\Omega_{22}}{\partial z_2}, \frac{\partial\Omega_{33}}{\partial z_3}\right)$$

respectively, whose coefficients are identical; it is therefore necessary for a maximum or minimum that

$$\begin{vmatrix} \frac{\partial\Omega_{11}}{\partial x_1}, & \frac{\partial\Omega_{22}}{\partial x_2}, & \frac{\partial\Omega_{33}}{\partial x_3} \\ \frac{\partial\Omega_{11}}{\partial y_1}, & \frac{\partial\Omega_{22}}{\partial y_2}, & \frac{\partial\Omega_{33}}{\partial y_3} \\ \frac{\partial\Omega_{11}}{\partial z_1}, & \frac{\partial\Omega_{22}}{\partial z_2}, & \frac{\partial\Omega_{33}}{\partial z_3} \end{vmatrix} = 0;$$

with the special form (1) of Ω the constituents of either row of this determinant are the same linear homogeneous functions of (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively, hence the determinant is the product of a constant determinant and

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_2 \\ z_1, & z_2, & z_3 \end{vmatrix},$$

hence we may write

$$x_3 = \lambda x_1 + \mu x_2, \quad y_3 = \lambda y_1 + \mu y_2, \quad z_3 = \lambda z_1 + \mu z_2,$$

and these substituted in the above-mentioned equations give

$$\lambda + \mu = 1,$$

i. e. the point (x_3, y_3, z_3) lies on the junction of (x_1, y_1, z_1) and (x_2, y_2, z_2). The same method gives the condition for a plane moving by the shortest course from one position to another. This method applied to the case of the variable straight line $L_3(a_3, b_3, c_3, f_3, g_3, h_3)$ moving by the shortest course from $L_1(a_1, b_1, c_1, f_1, g_1, h_1)$ to $L_2(a_2, b_2, c_2, f_2, g_2, h_2)$ gives

$$\begin{vmatrix} a_1, & b_1, & c_1, & -f_1, & -g_1, & -h_1 \\ f_1, & g_1, & h_1, & a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2, & -f_2, & -g_2, & -h_2 \\ f_2, & g_2, & h_2, & a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3, & -f_3, & -g_3, & -h_3 \\ f_3, & g_3, & h_3, & a_3, & b_3, & c_3 \end{vmatrix} = 0,$$

where also
$$a_3 f_3 + b_3 g_3 + c_3 h_3 = 0,$$
and we may write
$$\begin{aligned} a_1\alpha + b_1\beta + c_1\gamma - f_1\phi - g_1\chi - h_1\psi &= 0, \\ f_1\alpha + g_1\beta + h_1\gamma + a_1\phi + b_1\chi + c_1\psi &= 0, \\ a_2\alpha + b_2\beta + c_2\gamma - f_2\phi - g_2\chi - h_2\psi &= 0, \\ f_2\alpha + g_2\beta + h_2\gamma + a_2\phi + b_2\chi + c_2\psi &= 0, \\ a_3\alpha + b_3\beta + c_3\gamma - f_3\phi - g_3\chi - h_3\psi &= 0, \\ f_3\alpha + g_3\beta + h_3\gamma + a_3\phi + b_3\chi + c_3\psi &= 0; \end{aligned}$$
and it will be found that the conditions for maximum or minimum will be satisfied when
$$\alpha\phi + \beta\chi + \gamma\psi = 0,$$
i. e. when $\alpha, \beta, \gamma, \phi, \chi, \psi$ are the coordinates of a straight line, which, as is evident from the equations just written, intersects L_1, and its polar with respect to the absolute, and L_2 and its polar, and L_3 and its polar. This line must then be one of the two straight lines which meets each of the lines L_1, L_2 and their polars, and the conditions show that it may be either. Hence, in moving by the shortest course from L_1 to L_2, the variable line L_3 and its polar continually intersect both the straight lines intersecting L_1, L_2 and both their polars. This may be stated, as will be shown below, in other words, thus: *in passing by the shortest course from L_1 to L_2 the variable straight line L_3 slides along the two common perpendiculars to L_1 and L_2, and is itself constantly perpendicular to these common perpendiculars.*

I now find the conditions for zero-distance or "*parallelism.*" By (3), two points T_1 and T_2 are parallel when

(16) $$\Omega_{11}\Omega_{22} - \Omega_{12}^2 = 0,$$

i. e. when their junction is tangent to the absolute.

By (4), two planes M_1 and M_2 are parallel when

(17) $$\mho_{11}\mho_{22} - \mho_{12}^2 = 0,$$

i. e. when their intersection is tangent to the absolute.

By (5), a point and a plane are parallel when the point lies in the plane, *i. e.* when

(18) $$M_1 = 0.$$

By (6), a point T_1 and a straight line given as the junction of T_2 and T_3 are parallel when

(19) $$\begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{vmatrix} = 0,$$

i. e. when the plane of the point and line is tangent to the absolute. The same condition can be obtained more directly from (7), which gives as the condition that the point T_1 shall be parallel to the intersection L of the planes M_2 and M_3, where M_2 is the plane containing T_1 and L, and M_3 is any *other* plane through L_2,

(20) $$T_3 \sqrt{\mho_{22}} = 0,$$

where T_3 is the result of substituting the coordinates of M_3 in the (tangential) equation of T_1. Hence either T_1 lies on L, or their common plane is tangent to the absolute. The former is only a special case of the later when differently stated, thus: a point and line are parallel when it is possible to pass through them both a plane tangent to the absolute.

Similarly a plane M_1 and a line L given as the intersection of two planes M_2, M_3 are parallel, by (8), when

(21) $$\begin{vmatrix} \mho_{11} & \mho_{12} & \mho_{13} \\ \mho_{21} & \mho_{22} & \mho_{23} \\ \mho_{31} & \mho_{32} & \mho_{33} \end{vmatrix} = 0,$$

or, if the line L is given as the junction of T_2 and T_3, where T_2 is the intersection of M_1 and L,

(22) $$M_3 \sqrt{\Omega_{22}} = 0,$$

where M_3 is the result of substituting the coordinates of T_3 in the equation of M_1; *i. e.* the line and plane will be parallel when they intersect on the absolute, or when the line lies in the plane, in which latter case they have two common points on the absolute. It may be remarked that the condition $M_3 = 0$ is equivalent to the condition

$$AA' + BB' + CC' = 0$$

that the line

$$\frac{x - x'}{A'} = \frac{y - y'}{B'} = \frac{z - z'}{C'}$$

shall be parallel *in the Euclidean sense* to the plane

$$Ax + By + Cz + D = 0,$$

but in the non-Euclidean sense it is the condition that the line shall lie wholly in the plane.

By (11) the two straight lines L and L' are parallel only when $A=0$, or $C=0$, or $\Delta=0$; if $A=0$, or $C=0$, the distance $\overline{LL'}$ is indeterminate, and the lines cannot strictly be called parallel, indeed either of these conditions affects only one of the lines, it is the condition that the line in question shall be tangent to the absolute; the condition $\Delta=0$, *i. e.* that the lines intersect, makes one of the distances $\overline{LL'}=0$, the other value of $\overline{LL'}$ is then $2ik\cos^{-1}\frac{B}{\sqrt{AC}}$, and this also becomes 0 when $AC-B^2=0$, which is therefore the condition that L and L' shall be parallel in the strictest sense; and I shall call them parallel only when the two conditions are satisfied:

$$\Delta=0,\quad AC-B^2=0, \tag{23}$$

i. e. when the lines meet on the absolute.

I pass now to the conditions for the perpendicularity of two elements. It is convenient to have a name for the circular-function involved in any distance, and I call it the "*arc*" of that distance, thus from (3), etc.

$$\text{arc } \overline{TT'}=\cos^{-1}\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right), \text{ etc.}$$

Writing also co. = complement, it is easily seen that if E and E' are any two elements (of the same or different species) and if P is the polar of E (namely, P is plane, point, or straight line according as E is point, plane, or straight line),

$$\text{arc } \overline{EE'}=\frac{\partial}{2}-\text{arc } \overline{E'P}=\text{co. arc } \overline{E'P}, \tag{24}$$

provided the proper signs are given to the radicals involved.

We may also speak of the anharmonic ratio involved in the measure of a distance as the "*argument*" of that distance (abbreviated arg.), thus

$$\text{arg. } \overline{TT'}=\frac{\Omega_{12}+\sqrt{\Omega_{12}^2-\Omega_{11}\Omega_{22}}}{\Omega_{12}-\sqrt{\Omega_{12}^2-\Omega_{11}\Omega_{22}}};$$

then (24) may also be written

$$\text{arg. } \overline{EE'}=-\frac{1}{\text{arg. } \overline{E'P}}. \tag{25}$$

It is evident that the two elements E, E' are parallel when

$$\text{arc } \overline{EE'}=0 \text{ or } \partial,\ \textit{i. e. } \text{arg. } \overline{EE'}=1. \tag{26}$$

I call two elements E, E' perpendicular when

$$\text{arc } \overline{EE'}=\frac{\partial}{2},\ \textit{i. e. } \text{arg. } \overline{EE'}=-1. \tag{27}$$

The condition that two points $T(x_1, y_1, z_1)$ and $T'(x_2, y_2, z_2)$ shall be perpendicular is then

$$\Omega_{12} = 0, \tag{28}$$

i. e. either lies on the polar plane of the other with respect to the absolute.

Two planes $M(u_1, v_1, w_1)$ and $M'(u_2, v_2, w_2)$ are perpendicular when

$$\mho_{12} = 0, \tag{29}$$

i. e. either passes through the pole of the other with respect to the absolute.

A point $T(x_1, y_1, z_1)$ and a plane $M(u_1, v_1, w_1)$ are perpendicular when, using the notation of (5),

$$\Omega_{11}\mho_{11} - M_1^2 = 0, \tag{30}$$

i. e. the point T is parallel to (at a zero-distance from) the pole of M, and M is parallel to the polar plane of T.

A point T and a line L are perpendicular when, with the notation of (6),

$$\begin{cases} \Omega_{12}^2\Omega_{33} - 2\Omega_{12}\Omega_{13}\Omega_{23} + \Omega_{13}^2\Omega_{22} = 0, \text{ or} \\ R'^2\Omega_{12}^2 - 2N_1'N_2'\Omega_{12} + N_1'^2\Omega_{22} = 0, \text{ or} \\ \begin{vmatrix} \Omega_{11} & T_2 & N_T'' \\ T_2 & \mho_{22} & N_2'' \\ N_T'' & N_2'' & R''^2 \end{vmatrix} = 0, \text{ or} \begin{vmatrix} \Omega_{11} & T_2 & T_3 \\ T_2 & \mho_{22} & \mho_{23} \\ T_3 & \mho_{32} & \mho_{33} \end{vmatrix} = 0; \end{cases} \tag{31}$$

or, if (u_2, v_2, w_2) is the plane of T and L,

$$\begin{cases} \Omega_{11}(\mho_{22}\mho_{33} - \mho_{23}^2) - T_3^2\mho_{22} = 0, \text{ or} \\ \Omega_{11}(R''^2\mho_{22} - N_2''^2) - N_T''^2\mho_{22} = 0; \end{cases} \tag{32}$$

i. e. L is parallel to the polar plane of T, and T is parallel to the polar line of L.

A plane M and a line L are perpendicular when, with the notation of (9),

$$\begin{cases} \mho_{12}^2\mho_{33} - 2\mho_{12}\mho_{13}\mho_{23} + \mho_{13}^2\mho_{22} = 0, \text{ or} \\ R''^2\mho_{12}^2 - 2N_1''N_2''\mho_{12} + N_1''^2\mho_{22} = 0, \text{ or} \\ \begin{vmatrix} \mho_{11} & M_2 & N_M' \\ M_2 & \Omega_{22} & N_2' \\ N_M' & N_2' & R'^2 \end{vmatrix} = 0, \text{ or} \begin{vmatrix} \mho_{11} & M_2 & M_3 \\ M_2 & \Omega_{22} & \Omega_{23} \\ M_3 & \Omega_{32} & \Omega_{33} \end{vmatrix} = 0; \end{cases} \tag{33}$$

or, if (x_2, y_2, z_2) is the intersection of M and L,

$$\begin{cases} \mho_{11}(\Omega_{22}\Omega_{33} - \Omega_{23}^2) - M_3^2\Omega_{22} = 0, \text{ or} \\ \mho_{11}(R'^2\Omega_{22} - N_2'^2) - N_M'^2\Omega_{22} = 0; \end{cases} \tag{34}$$

i. e. M is parallel to the polar line of L, and L is parallel to the pole of M with respect to the absolute.

In the case of two lines L, L', with the notation of (11), one value of arc $\overline{LL'}$ is $\frac{\partial}{2}$ when $B = 0$, and both values are $\frac{\partial}{2}$ when $B = 0$ and $AC + \Delta^2 = 0$. In the former case I call the lines simply "*perpendicular,*" and in the latter case they may be called "*doubly perpendicular.*" The condition for the perpendicularity of two lines is then

(35) $$B=0 \ [\text{or } \mathfrak{g}=0, \text{ or } \mathfrak{D}=0].$$

This condition is equivalent to

(36) $$\Omega_{13}+\lambda\Omega_{23}=0 \text{ and } \Omega_{14}+\lambda\Omega_{24}=0, \text{ or}$$

(37) $$\Omega_{13}+\mu\Omega_{14}=0 \text{ and } \Omega_{23}+\mu\Omega_{24}=0,$$

for some values of λ and μ. The equations (36) imply that a certain point $\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right)$ of L lies on the polar of L', and equations (37) imply that a certain point $\left(\frac{x_3+\mu x_4}{1+\mu}, \frac{y_3+\mu y_4}{1+\mu}, \frac{z_3+\mu z_4}{1+\mu}\right)$ of L' lies on the polar of L, *i. e.* L intersects the polar line of L', and L' intersects the polar line of L.

[The condition for double perpendicularity of L and L' is

(38) $$B=0, \text{ and } AC+\Delta^2=0;$$

now, by (10), these two conditions give

$$\alpha\alpha'+\beta\beta'+\gamma\gamma'+\delta\delta'=0,$$

and by (36) and (37) and the former definitions of α, β, γ, δ, α', β', γ', δ',

$$\alpha=(\Omega_{11}+\lambda\Omega_{12})\Omega_{23}, \quad \alpha'=\Omega_{23}\Omega_{44}-\Omega_{24}\Omega_{34}=-(\Omega_{34}+\mu\Omega_{44})\Omega_{24},$$
$$\beta=(\Omega_{11}+\lambda\Omega_{12})\Omega_{24}, \quad \beta'=\Omega_{24}\Omega_{33}-\Omega_{23}\Omega_{34}=(\Omega_{33}+\mu\Omega_{34})\Omega_{24},$$
$$\gamma=(\Omega_{12}+\lambda\Omega_{22})\Omega_{23}, \quad \gamma'=-\lambda(\Omega_{24}\Omega_{34}-\Omega_{23}\Omega_{44})=\lambda\alpha',$$
$$\delta=(\Omega_{12}+\lambda\Omega_{22})\Omega_{24}, \quad \delta'=-\lambda(\Omega_{23}\Omega_{34}-\Omega_{24}\Omega_{33})=\lambda\beta',$$
$$\alpha\alpha'+\beta\beta'+\gamma\gamma'+\delta\delta'=(\Omega_{11}+2\lambda\Omega_{12}+\lambda^2\Omega_{22})(\Omega_{23}\alpha'+\Omega_{24}\beta')$$
$$=\Omega_{24}^2(\Omega_{11}+2\lambda\Omega_{12}+\lambda^2\Omega_{22})(\Omega_{33}+2\mu\Omega_{34}+\mu^2\Omega_{44}),$$

i. e. either $\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right)$ or $\left(\frac{x_3+\mu x_4}{1+\mu}, \frac{y_3+\mu y_4}{1+\mu}, \frac{z_3+\mu z_4}{1+\mu}\right)$ lies on the absolute, and hence at least one of the lines L, L' is parallel to the polar of the other.]

If L and L' are simply perpendicular and intersect, $B=0$ and $\Delta=0$, one value of arc $\overline{LL'}$ is $\frac{1}{2}\,\Theta$, while the other is 0.

From these definitions of perpendicularity it follows that *the arc of the distance between two given planes is equal to the arc of the (non-evanescent) distance between the two straight lines drawn from any point of their intersection, one in each plane perpendicular to their intersection.* This is easily proved; let $M(u_1, v_1, w_1)$ and $M'(u_2, v_2, w_2)$ be the two planes, and let $M''(u_3, v_3, w_3)$ be the plane of the two perpendiculars to which reference is made in the theorem; then, since the intersections of *M* and *M''* and of *M'* and *M''* are perpendicular to the intersection of *M* and *M'*,

$$\mho_{11}\mho_{23} - \mho_{12}\mho_{13} = 0,$$
$$\mho_{12}\mho_{23} - \mho_{22}\mho_{13} = 0,$$

by (35), but in general $\mho_{11}\mho_{22} - \mho_{12}^2$ does not vanish, hence

$$\mho_{13} = 0, \quad \mho_{23} = 0,$$

and the angle between *MM''* and *M'M''* is, by (12),

$$2ik^{\text{v}} \cos^{-1} \frac{\mathfrak{V}_{12}}{\sqrt{\mathfrak{V}_{11}\mathfrak{V}_{22}}},$$

which proves the theorem.

This proof also shows that *if two planes are perpendicular to the same third plane, their intersection is perpendicular to that plane and to every straight line in it.* Similarly *if two points are perpendicular to the same third point, their junction is perpendicular to that point and to every straight line through it.* With these definitions it is not true that every straight line perpendicular to a given plane is perpendicular to every straight line in that plane, or even to every intersecting line in that plane; through any given point there pass an infinite number of straight lines perpendicular to a given plane, namely all which pass through the point and lie in either of the planes tangent to the absolute through the junction of the point with the pole of the given plane. One of these lines, namely the junction of the given point with the pole of the given plane is perpendicular to every straight line in the plane. [It is possible that the definition of perpendicularity of a point and a plane, a point and a line, and of a plane and a line should be narrower than that here given; namely, it may be that a point should be considered perpendicular to a given plane only when it *coincides* with the pole of the plane, instead of being simply at a zero-distance from it; that a straight line should be considered perpendicular to a point only when it lies in the polar plane of the point; and that a straight line should be considered perpendicular to a plane only when it passes through the pole of the plane. This narrower definition of perpendicularity will not, however, affect the further results of this paper.] If two straight lines are perpendicular to the same plane, they will not necessarily be parallel to each other; and two planes which are perpendicular to the same straight line will not necessarily be parallel to each other. It is also evident that two planes parallel to the same plane are not in general parallel to each other; and two straight lines parallel to the same straight line are not necessarily parallel to each other (the odds are equal that they will be so).

If $T(x, y, z)$, $T'(x+dx_1, y+dy_1, z+dz_1)$ and $T''(x+dx_2, y+dy_2, z+dz_2)$ are three infinitely near points, and on TT' and TT'' a plane quadrilateral is constructed whose other two sides are parallel to these, it will be seen, from (13), (12) and (6), that

$$\overline{TT'} = \frac{2ik}{\Omega}\sqrt{\Omega\delta_1^2\Omega-(\delta_1\Omega)^2} = \frac{2ik}{\Omega}\sqrt{A}, \quad \overline{TT''} = \frac{2ik}{\Omega}\sqrt{\Omega\delta_2^2\Omega-(\delta_2\Omega)^2} = \frac{2ik}{\Omega}\sqrt{C},$$

$$\sin\frac{\overline{(TT', TT'')}}{2ik^{\mathrm{V}}} = \sqrt{\frac{AC-B^2}{AC}}, \quad \overline{(TT', T'')} = 2ik''' \sqrt{\frac{\begin{vmatrix} \Omega, & \delta_1\Omega, & \delta_2\Omega \\ \delta_1\Omega, & \delta_1^2\Omega, & \delta_1\delta_2\Omega \\ \delta_2\Omega, & \delta_2\delta_1\Omega, & \delta_2^2\Omega \end{vmatrix}}{\Omega[\Omega\delta_1^2\Omega-(\delta_1\Omega)^2]}}$$

(dx_1, &c., being infinitesimal), where TT' and TT'' denote the lines TT' and TT'', $\overline{(TT', TT'')}$ is the angle between them, $\overline{(TT', T'')}$ is the distance between the line TT' and the point T'', $\delta_1\Omega$, $\delta_2\Omega$, $\delta_1^2\Omega$, $\delta_2^2\Omega$ are what $\delta\Omega$ and $\delta^2\Omega$ already defined become on the substitution of dx_1, dy_1, dz_1 and dx_2, dy_2, dz_2 for dx, dy, dz, and

$$\begin{aligned}\delta_1\delta_2\Omega = \delta_2\delta_1\Omega = \frac{1}{2}\Big[&\frac{\partial^2\Omega}{\partial x^2}dx_1dx_2 + \frac{\partial^2\Omega}{\partial y^2}dy_1dy_2 + \frac{\partial^2\Omega}{\partial z^2}dz_1dz_2 \\ &+ \frac{\partial^2\Omega}{\partial y\partial z}(dy_1dz_2+dz_1dy_2) + \frac{\partial^2\Omega}{\partial z\partial x}(dz_1dx_2+dx_1dz_2) \\ &+ \frac{\partial^2\Omega}{\partial x\partial y}(dx_1dy_2+dy_1dx_2)\Big],\end{aligned}$$

$$A = \Omega\delta_1^2\Omega-(\delta_1\Omega)^2, \quad C = \Omega\delta_2^2\Omega-(\delta_2\Omega)^2, \quad B = \Omega\delta_1\delta_2\Omega-\delta_1\Omega\delta_2\Omega,$$

$$\begin{aligned}B^2-AC &= \begin{vmatrix} \Omega\delta_1^2\Omega-(\delta_1\Omega)^2, & \Omega\delta_1\delta_2\Omega-\delta_1\Omega\delta_2\Omega \\ \Omega\delta_2\delta_1\Omega-\delta_2\Omega\delta_1\Omega, & \Omega\delta_2^2\Omega-(\delta_2\Omega)^2 \end{vmatrix} \\ &= \Omega\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix};\end{aligned}$$

hence

$$\overline{TT'}\;\overline{TT''}\sin\frac{\overline{(TT', TT'')}}{2ik^{\mathrm{V}}} = \frac{(2ik)^2}{\Omega^{\frac{3}{2}}}\sqrt{\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}},$$

$$\overline{TT''}\,\overline{(TT'', T')} = \overline{TT'}\,\overline{(TT', T'')} = \frac{(2ik)(2ik''')}{\Omega^{\frac{3}{2}}}\sqrt{\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}};$$

I therefore take as the measure of the area of the infinitesimal parallelogram, k^{VI} being a new constant,

$$(39)\qquad \overline{TT'\,T''} = d\bar{\sigma} = \frac{(2ik^{\mathrm{VI}})^2}{\Omega^{\frac{3}{2}}}\sqrt{\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}} = \frac{1}{K_2\,\Omega^{\frac{3}{2}}}\sqrt{\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}};$$

where $K_2 = \frac{1}{(2ik^{\mathrm{VI}})^2}$, and this expression for $d\bar{\sigma}$ is not altered if T'' be moved to any other position on the side opposite TT', provided its distance from T remain infinitesimal; *the areas of infinitesimal parallelograms are therefore proportional to the products of their bases by their altitudes*, either side being taken as the base. The area of any figure is then measured by the integral of (39) over the whole figure; namely, the figure is to be considered as divided into infinitesimal parallelograms, and its area is the sum of the areas of these parallelograms. It is to be noticed that the expression in the right-hand member of (39) remains unaltered if T'' be infinitely little displaced on any straight line, through its former position, intersecting TT' at a distance from T which is not infinitesimal; hence for this purpose an infinitesimal parallelogram may be defined as an infinitesimal quadrilateral whose opposite sides intersect at distances from its vertices which are not infinitesimal.

Similarly if $T'''\,(x + dx_3,\ y + dy_3,\ z + dz_3)$ be a fourth point infinitely near T,

$$\overline{TT'''} = \frac{2ik}{\Omega}\sqrt{\Omega\delta_3^2\Omega - (\delta_3\Omega)^2},$$

where the notation is sufficiently explained by the above definitions of δ and δ^2; the equation of the plane $TT'\,T''$ (ξ, η, ζ being current coordinates) is

$$\begin{vmatrix} \xi & \eta & \zeta & 1 \\ x & y & z & 1 \\ dx_1 & dy_1 & dz_1 & 0 \\ dx_2 & dy_2 & dz_2 & 0 \end{vmatrix} = 0,$$

or

$$u\xi + v\eta + w\zeta + 1 = 0,$$

where

$$\begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix} u = -\begin{vmatrix} dy_1 & dz_1 \\ dy_2 & dz_2 \end{vmatrix}, \quad \begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix} v = -\begin{vmatrix} dz_1 & dx_1 \\ dz_2 & dx_2 \end{vmatrix},$$

$$\begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix} w = -\begin{vmatrix} dx_1 & dy_1 \\ dx_2 & dy_2 \end{vmatrix},$$

whence

$$\begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix}^2 \mho = \begin{vmatrix} dy_1 & dz_1 \\ dy_2 & dz_2 \end{vmatrix}^2 + \begin{vmatrix} dz_1 & dx_1 \\ dz_2 & dx_2 \end{vmatrix}^2 + \begin{vmatrix} dx_1 & dy_1 \\ dx_2 & dy_2 \end{vmatrix}^2 - \begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix}^2$$

$$= \delta_1^2\Omega\,\delta_2^2\Omega - (\delta_1\delta_2\Omega)^2 - \begin{vmatrix} \Omega+1 & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix}$$

$$= - \begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix};$$

then, with the notation of (9),

$$\begin{vmatrix} x & y & z \\ dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \end{vmatrix} M_3 = - \begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix},$$

$$\sin\frac{\overline{(TT'\,T'',\ TT''')}}{2ik^{\mathrm{IV}}} = \frac{\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix} \sqrt{\Omega}}{\sqrt{-\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix} [\Omega\delta_3^2\Omega - (\delta_3\Omega)^2]}},$$

where $TT'\,T''$ means the plane of T, T' and T'', and by (5), replacing the infinitesimal angle by its sine,

$$\overline{TT'\,T'',\ T'''} = 2ik'' \frac{\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix}}{\sqrt{-\begin{vmatrix} \Omega & \delta_1\Omega & \delta_2\Omega \\ \delta_1\Omega & \delta_1^2\Omega & \delta_1\delta_2\Omega \\ \delta_2\Omega & \delta_2\delta_1\Omega & \delta_2^2\Omega \end{vmatrix} \Omega}};$$

hence

$$\overline{TT'\,T''}\ \overline{TT'''} \sin\frac{\overline{(TT'\,T'',\ TT''')}}{2ik^{\mathrm{IV}}} = \frac{(2ik)(2ik^{\mathrm{VI}})^2}{i\Omega^2} \begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix},$$

$$\overline{TT'\,T''}\ \overline{TT'\,T'',\ T'''} = \frac{(2ik'')(2ik^{\mathrm{VI}})^2}{i\Omega^2} \begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix};.$$

I take therefore as the measure of the volume of the infinitesimal parallelopiped on the three edges LL', LL'' and LL''', k^{VI} being a new constant,

$$(40)\qquad \overline{TT'T''T'''} = d\bar{v} = \frac{(2ik^{\text{VII}})^3}{i\Omega^2}\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix} = \frac{1}{iK_3\Omega^2}\begin{vmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{vmatrix},$$

where $K_3 = \frac{1}{(2ik^{\text{VII}})^3}$, and an infinitesimal parallelopiped need merely be defined as an infinitesimal hexahedron whose opposite faces meet at distances from its vertices which are not infinitesimal. It then follows that *the volumes of infinitesimal parallelopipeds are proportional to the products of their bases by their altitudes,* either face of each being taken as the base. The volume of any solid may be found by integrating (40) over the whole solid; and for this purpose we may suppose the solid divided into infinitesimal parallelopipeds by planes whose equations are $x = \text{const.}$, $y = \text{const.}$, $z = \text{const.}$, so that dy_1, dz_1, dx_2, dz_2, dx_3, dy_3 vanish, and

$$(41)\qquad d\bar{v} = \frac{1}{iK_3}\frac{dx_1\,dy_2\,dz_3}{\Omega^2}.$$

The constants K_2 and K_3 correspond to the constant curvatures of space of two and three dimensions in Gauss' theory (see p. 210).

A convenient transformation is

$$(42)\qquad x = \rho\sin\vartheta\cos\omega,\quad y = \rho\sin\vartheta\sin\omega,\quad z = \rho\cos\vartheta,$$

whence

$$(43)\qquad \begin{cases} dx = d\rho\sin\vartheta\cos\omega + \rho\cos\vartheta\cos\omega\,d\vartheta - \rho\sin\vartheta\sin\omega\,d\omega, \\ dy = d\rho\sin\vartheta\sin\omega + \rho\cos\vartheta\sin\omega\,d\vartheta + \rho\sin\vartheta\cos\omega\,d\omega, \\ dz = d\rho\cos\vartheta \qquad - \rho\sin\vartheta\,d\vartheta; \end{cases}$$

then, with the value of Ω given by (1),

$$\delta\Omega = \rho\,d\rho,\quad \delta^2\Omega = d\rho^2 + \rho^2 d\vartheta^2 + \rho^2\sin^2\vartheta . d\omega^2,$$

$$\delta_1\delta_2\Omega = d\rho_1 d\rho_2 + \rho^2 d\vartheta_1 d\vartheta_2 + \rho^2\sin^2\vartheta . d\omega_1 d\omega_2,$$

and (13), (39) and (40) become

$$(44)\qquad d\bar{s} = 2ik\frac{\sqrt{-d\rho^2 - \rho^2(1-\rho^2)(d\vartheta^2 + \sin^2\vartheta\,d\omega^2)}}{1-\rho^2},$$

$$(45)\quad d\bar{\sigma} = \frac{1}{K_2}\frac{\rho}{1-\rho^2}\sqrt{\frac{\rho^2(1-\rho^2)\sin^2\vartheta(d\vartheta_1 d\omega_2 - d\vartheta_2 d\omega_1)^2 + (d\rho_1 d\vartheta_2 - d\rho_2 d\vartheta_1)^2 + \sin^2\vartheta(d\rho_1 d\omega_2 - d\rho_2 d\omega_1)^2}{1-\rho^2}},$$

$$(46)\qquad d\bar{v} = \frac{1}{iK_3}\frac{\rho^2\sin\vartheta}{(1-\rho^2)^2}\begin{vmatrix} d\rho_1 & d\vartheta_1 & d\omega_1 \\ d\rho_2 & d\vartheta_2 & d\omega_2 \\ d\rho_3 & d\vartheta_3 & d\omega_3 \end{vmatrix},$$

which expressions may be taken either positive or negative; and, if a solid is divided into infinitesimal parallelopipeds by surfaces of the systems $\rho = \text{const.}$, $\vartheta = \text{const.}$, $\omega = \text{const.}$, the volume of any one of these parallelopipeds is

$$d\bar{v} = \frac{1}{iK_3}\frac{\rho^2 \sin\vartheta\, d\rho_1\, d\vartheta_2\, d\omega_3}{(1-\rho^2)^2}, \tag{47}$$

corresponding to (41).

It will be seen that the new definitions here given are extensions of Professor Cayley's definitions of "projective" measurement (Sixth Memoir upon Quantics, *Phil. Trans.*, vol. 149, 1859) as generalized by Professor Klein (*Math. Ann.*, Vol. IV).

Application to the Sphere.

A sphere is defined as the locus of points at a given distance (radius of the sphere) from a given point (centre of the sphere); the radius being $\bar{r}$ and centre (x_1, y_1, z_1), the condition that (x_2, y_2, z_2) should be a point of the sphere is, by (3),

$$\Omega_{12}^2 = \Omega_{11}\Omega_{22}\cos^2\left(\frac{\bar{r}}{2ik}\right), \tag{48}$$

or, say,

$$\lambda\Omega_{11}\Omega_{22} - \mu\Omega_{12}^2 = 0, \tag{49}$$

where Ω_{11}, Ω_{22}, Ω_{12} are defined by (1).

The sphere (49) will be a cone for certain values of λ, μ determined by the equation

$$0 = \begin{vmatrix} \lambda\Omega_{11} - \mu x_1^2\,, & -\mu x_1 y_1\,, & -\mu x_1 z_1\,, & \mu x_1 \\ -\mu x_1 y_1\,, & \lambda\Omega_{11} - \mu y_1^2\,, & -\mu y_1 z_1\,, & \mu y_1 \\ -\mu x_1 z_1\,, & -\mu y_1 z_1\,, & \lambda\Omega_{11} - \mu z_1^2\,, & \mu z_1 \\ \mu x_1\,, & \mu y_1\,, & \mu z_1\,, & -\lambda\Omega_{11} - \mu \end{vmatrix} = \lambda^3(\mu - \lambda)\Omega_{11}^4,$$

i. e. in general for $\lambda = 0$ and $\mu = \lambda$.

For $\lambda = 0$ the equation of the sphere is

$$\Omega_{12}^2 = 0, \tag{50}$$

i. e. the sphere consists of two planes coincident with the polar plane of the centre with respect to the absolute. In this case

$$\cos^2\left(\frac{\bar{r}}{2ik}\right) = \frac{\lambda}{\mu} = 0,$$

i. e.

$$\bar{r} = ik\,\partial. \tag{51}$$

A great circle on such a sphere consists of two coincident straight lines. It is then evident that a plane may be considered as *half* a sphere of radius $ik\partial$, and a straight line as *half* a circle of the same radius. The outside of such a sphere is the portion of space included between the coincident planes, so that the sphere *includes* all space, and the outside of such a circle is the portion of its plane between the coincident lines, so that the circle *includes* the whole plane. I shall presently find, by this means, the length of the whole straight line, the circumference and area of the whole plane, the area of the boundary of all space, and the volume of all space.

For $\mu = \lambda$ the equation of the sphere is

(52) $$\Omega_{11}\Omega_{22} - \Omega_{12}^2 = 0,$$

i. e. the sphere is the tangent cone to the absolute from the centre. In this case

$$\cos^2\left(\frac{\bar{r}}{2ik}\right) = \frac{\lambda}{\mu} = 1,$$

i. e.

(53) $$\bar{r} = 0.$$

In the general consideration of the sphere, I take, for simplicity, the centre as origin, and write the variables (x, y, z) instead of (x_2, y_2, z_2); equation (48) then becomes

$$\Omega = -\sec^2\left(\frac{\bar{r}}{2ik}\right),$$

i. e.

(54) $$x^2 + y^2 + z^2 = 1 - \sec^2\left(\frac{\bar{r}}{2ik}\right) = -\tan^2\left(\frac{\bar{r}}{2ik}\right) = \tau^2, \text{ say.}$$

With the polar system (42), the equation of the sphere becomes

(55) $$\rho = \tau.$$

The form (1) of Ω allows perfect freedom in the choice of the origin, of one coordinate axis through the origin, and of one coordinate plane through this axis, but when the origin, axis and plane have been chosen, the whole system is determined. The circumference and area of any great circle of the sphere will be most conveniently determined by taking this plane as the plane $z = 0$, *i. e.* the plane $\vartheta = \frac{1}{2}\eth$. Then the circumference of the circle is, by (44) (putting $\rho = \tau$, $\vartheta = \frac{1}{2}\eth$, $d\rho = 0$, $d\vartheta = 0$),

(56) $$C_{\bar{r}} = \pm 2ik\frac{\tau i}{\sqrt{1-\tau^2}}\int_0^{2\eth} d\omega = \pm 4ik\eth\,\frac{\tau i}{\sqrt{1-\tau^2}} = \pm 4ik\eth\,\sin\left(\frac{\bar{r}}{2ik}\right)$$
$$= \pm 2k\eth\left(\text{ϭ}^{\frac{\bar{r}}{2k}} - \text{ϭ}^{-\frac{\bar{r}}{2k}}\right),$$

the latter of which expressions was given by Gauss; and the area of the circle is, by (45) (putting $\vartheta = \frac{1}{2}\eth$, $d\rho_1 = d\rho$, $d\vartheta_1 = 0$, $d\omega_1 = 0$, $d\rho_2 = 0$, $d\vartheta_2 = 0$, $d\omega_2 = d\omega$),

$$(57)\qquad A_{\bar r} = \pm\frac{1}{K_2}\int_{0\;\rho}^{\tau}\int_{0\;\omega}^{2\pi}\frac{\rho d\rho d\omega}{(1-\rho^2)^{\frac{3}{2}}} = \pm\frac{2\pi}{K_2}\Big(\frac{1}{\sqrt{1-\tau^2}}-1\Big)$$

$$= \mp\frac{2\pi}{K_2}\Big[1-\cos\Big(\frac{\bar r}{2ik}\Big)\Big] = \mp\frac{4\pi}{K_2}\sin^2\Big(\frac{\bar r}{4ik}\Big) = \pm\frac{\pi}{K_2}\Big(e^{\frac{\bar r}{4k}} - e^{-\frac{\bar r}{4k}}\Big)^2.$$

The area of the surface of the sphere of radius $\bar r$ is, by (45) (putting $\rho=\tau$, $d\rho_1=0$, $d\vartheta_1=d\vartheta$, $d\omega_1=0$, $d\rho_2=0$, $d\vartheta_2=0$, $d\omega_2=d\omega$)

$$(58)\qquad S_{\bar r} = \pm\frac{1}{K_2}\frac{\tau^2}{1-\tau^2}\int_{0\;\vartheta}^{\pi}\int_{0\;\omega}^{2\pi}\sin\vartheta\, d\vartheta\, d\omega = \mp\frac{4\pi}{K_2}\sin^2\Big(\frac{\bar r}{2ik}\Big)$$

$$= \pm\frac{\pi}{K_2}\Big(e^{\frac{\bar r}{2k}} - e^{-\frac{\bar r}{2k}}\Big)^2;$$

and the volume of the sphere is, by (47) (putting $d\rho_1=d\rho$, $d\vartheta_2=d\vartheta$, $d\omega_3=d\omega$),

$$(59)\qquad V_{\bar r} = \pm\frac{1}{iK_3}\int_{0\;\rho}^{\tau}\int_{0\;\vartheta}^{\pi}\int_{0\;\omega}^{2\pi}\frac{\rho^2\sin\vartheta\, d\rho\, d\vartheta\, d\omega}{(1-\rho^2)^2} = \pm\frac{\pi}{K_3}\Big[\frac{\bar r}{ik} - \sin\Big(\frac{\bar r}{ik}\Big)\Big]$$

$$= \mp\frac{\pi}{2iK_3}\Big(e^{\frac{\bar r}{k}} - e^{-\frac{\bar r}{k}} - 2\frac{\bar r}{k}\Big).$$

From (57) and (58) follows

$$(60)\qquad S_{\bar r} = A_{2\bar r},$$

i. e. the area of the surface of a sphere equals the area of a circle of twice the radius, which corresponds to the theorem of the Euclidean geometry, commonly stated thus: *the area of the surface of a sphere equals four times the area of a circle of the same radius.*

Formulæ (56), (57), (58) and (59) applied to the case in which $\bar r = ik\pi$ give

$$(61)\qquad C_{ik\pi} = 4ik\pi,\; A_{ik\pi} = \frac{2\pi}{K_2},\; S_{ik\pi} = \frac{4\pi}{K_2},\; V_{ik\pi} = \frac{\pi^2}{K_3},$$

whence *the length of the whole straight line is* $2ik\pi$, *the area of the whole plane is* $\frac{2\pi}{K_2}$, *the area of the surface of all space is* $\frac{4\pi}{K_2}$, *and the volume of all space is* $\frac{\pi^2}{K_3}$.

In the Euclidean geometry $K_2 = -\frac{1}{4k^2}$, $K_3 = \frac{i}{8k^3}$, and k is infinite (see Professor Klein's paper, *Math. Annalen*, Vol. IV, p. 592, lines 3, 4, where c has the same meaning as k in the present paper), hence $\frac{\bar r}{ik}$ is infinitesimal,

$$\sin\Big(\frac{\bar r}{2ik}\Big) = \frac{\bar r}{2ik},\; \sin\Big(\frac{\bar r}{4ik}\Big) = \frac{\bar r}{4ik},\; \frac{\bar r}{ik} - \sin\Big(\frac{\bar r}{ik}\Big) = \frac{1}{6}\Big(\frac{\bar r}{ik}\Big)^3, \text{ and}$$

$$(62)\qquad C_{\bar r} = 2\pi\bar r,\; A_{\bar r} = \pi\bar r^2,\; S_{\bar r} = 4\pi\bar r^2,\; V_{\bar r} = \frac{4}{3}\pi\bar r^3,$$

the ordinary formulæ.

Before considering the areas of figures drawn on a sphere, it will be convenient to obtain the formulæ of non-Euclidean spherical trigonometry. A spherical triangle is a figure on the sphere bounded by three arcs of great circles (intersections of the sphere with planes through the centre). The angle between any two such arcs may be measured by the distance (angle) between their planes or by the distance (angle) between the tangents to them at their intersection; the angles of the triangle measured in the second way I call $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, then it may easily be proved that the angles measured in the first way will be $\frac{k'}{k^{\mathrm{V}}}\bar{\alpha}$, $\frac{k'}{k^{\mathrm{V}}}\bar{\beta}$, $\frac{k'}{k^{\mathrm{V}}}\bar{\gamma}$; the lengths of the sides or bounding arcs opposite $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ I call respectively $\bar{a}$, $\bar{b}$, $\bar{c}$.

I begin with the case of a right triangle (a right angle is the angle between perpendicular lines); let $\bar{\gamma}$ be the right angle, *i. e.* $\bar{\gamma} = ik^{\mathrm{V}}\Theta$, take the centre of the sphere as origin, the radius to the angle $\bar{\gamma}$ as axis of x, the plane of the side $\bar{a}$ as the plane $y = 0$ (*i. e.* $\omega = 0$), then the plane of the side $\bar{b}$ will be the plane $z = 0$ (*i. e.* $\vartheta = \frac{1}{2}\Theta$). For the radius to the angle $\bar{\beta}$, then, $\omega = 0$, $\vartheta = \vartheta_1$, say, and for the radius to the angle $\bar{\alpha}$, $\vartheta = \frac{1}{2}\Theta$, $\omega = \omega_1$, say. Then, by (44)

$$\bar{a} = \mp 2k\frac{\tau}{\sqrt{1-\tau^2}}\int\limits_{\vartheta_1}^{\frac{1}{2}\Theta} d\vartheta = \pm 2ik\sin\left(\frac{\bar{r}}{2ik}\right).\left(\tfrac{1}{2}\Theta - \vartheta_1\right),$$

$$\bar{b} = \mp 2k\frac{\tau}{\sqrt{1-\tau^2}}\int\limits_{0}^{\omega_1} d\omega = \pm 2ik\sin\left(\frac{\bar{r}}{2ik}\right).\omega_1;$$

the plane of $\bar{c}$, the hypothenuse of the triangle, has the equation

$$\begin{vmatrix} \sin\vartheta\cos\omega & \sin\vartheta\sin\omega & \cos\vartheta \\ \cos\omega_1 & \sin\omega_1 & 0 \\ \sin\vartheta_1 & 0 & \cos\vartheta_1 \end{vmatrix} = 0,$$

which may also be written

$$\sin(\omega_1 - \omega) = \tan\vartheta_1 \sin\omega_1 \cot\vartheta,$$

from which follows

$$\sqrt{d\vartheta^2 + \sin^2\vartheta.d\omega^2} = \frac{\sin\vartheta.d\vartheta\sqrt{\tan^2\vartheta_1\sin^2\omega_1 + 1}}{\sqrt{1-(\tan^2\vartheta_1\sin^2\omega_1+1)\cos^2\vartheta}},$$

and then, by (44),

$$\bar{c} = \mp 2k\frac{\tau}{\sqrt{1-\tau^2}}\int\limits_{\vartheta_1}^{\frac{1}{2}\Theta}{}_{\vartheta}\sqrt{d\vartheta^2+\sin^2\vartheta.d\omega^2} = \mp 2k\frac{\tau}{\sqrt{1-\tau^2}}\int\limits_{\vartheta_1}^{\frac{1}{2}\Theta}\frac{\sin\vartheta\, d\vartheta\sqrt{\tan^2\vartheta_1\sin^2\omega_1+1}}{\sqrt{1-(\tan^2\vartheta_1\sin^2\omega_1+1)\cos^2\vartheta}}$$

$$= \pm 2ik\sin\left(\frac{\bar{r}}{2ik}\right)\cos^{-1}(\sin\vartheta_1\cos\omega_1);$$

the planes of $\bar{a}$, $\bar{b}$, $\bar{c}$ are

$$y=0\,,\ z=0\,,\ -x\cos\vartheta_1\sin\omega_1+y\cos\vartheta_1\cos\omega_1+z\sin\vartheta_1\sin\omega_1=0\,,$$

respectively, hence by (4),

$$\frac{k'}{k^{\mathrm{V}}}\bar{\alpha}=2ik'\cos^{-1}\left(\frac{\sin\vartheta_1\sin\omega_1}{\sqrt{\cos^2\vartheta_1+\sin^2\vartheta_1\sin^2\omega_1}}\right)=2ik'\tan^{-1}(\cot\vartheta_1\operatorname{cosec}\omega_1),$$

$$\frac{k'}{k^{\mathrm{V}}}\bar{\beta}=2ik'\cos^{-1}\left(\frac{\cos\vartheta_1\cos\omega_1}{\sqrt{\cos^2\vartheta_1+\sin^2\vartheta_1\sin^2\omega_1}}\right)=2ik'\tan^{-1}(\tan\vartheta_1\tan\omega_1),$$

i. e. $\bar{\alpha}=2ik^{\mathrm{V}}\tan^{-1}(\cot\vartheta_1\operatorname{cosec}\omega_1)$, $\bar{\beta}=2ik^{\mathrm{V}}\tan^{-1}(\sec\vartheta_1\tan\omega_1)$.

From these values of $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{\alpha}$, $\bar{\beta}$ follow

$$\cos\left(\frac{\bar{c}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)=\cos\left(\frac{\bar{a}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)\cdot\cos\left(\frac{\bar{b}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right),$$

$$\tan\left(\frac{\bar{a}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)=\sin\left(\frac{\bar{b}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)\tan\left(\frac{\bar{\alpha}}{2ik^{\mathrm{V}}}\right),$$

$$\tan\left(\frac{\bar{b}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)=\sin\left(\frac{\bar{a}}{2ik\sin\left(\frac{\bar{r}}{2ik}\right)}\right)\tan\left(\frac{\bar{\beta}}{2ik^{\mathrm{V}}}\right),$$

where the ambiguous signs in the right-hand members have been so determined that for small values of $\bar{a}$ and $\bar{b}$, $\bar{c}$ shall be small, and for a small positive value of $\bar{b}$ and values of $\bar{\alpha}$ and $\bar{\beta}$ less than $ik^{\mathrm{V}}\Theta$, $\bar{a}$ shall have a small positive value. Now these three formulæ are what the corresponding formulæ of the Euclidean trigonometry for the sphere of radius unity become when for each side is substituted that side divided by $2ik\sin\left(\frac{\bar{r}}{2ik}\right)$ and for each angle is substituted that angle divided by $2ik^{\mathrm{V}}$; and it is possible to obtain the formulæ for any spherical triangle whatever, by dividing it into two right triangles, from the formulæ just written; hence *the formulæ for the non-Euclidean trigonometry on a sphere of radius* $\bar{r}$ *are obtained from the formulæ of the ordinary spherical trigonometry by substituting for each side that side divided by* $2ik\sin\left(\frac{\bar{r}}{2ik}\right)$ *and for each angle that angle divided by* $2ik^{\mathrm{V}}$.

The area of the right triangle above considered is, by (45) (putting $\rho=\tau$, $d\vartheta_1=d\vartheta$, $d\omega_1=0$, $d\vartheta_2=0$, $d\omega_2=d\omega$)

$$\bar{\sigma}=\pm\frac{1}{K_2}\frac{\tau^2}{1-\tau^2}\underset{0}{\int_{\omega}^{\omega_1}}\int_{\vartheta}^{\frac{1}{2}\Theta}\sin\vartheta\,d\vartheta\,d\omega$$

where the lower limit of ϑ is

$$\cot^{-1}\left(\frac{\sin(\omega_1-\omega)}{\tan\vartheta_1\sin\omega_1}\right)=\cos^{-1}\left(\frac{\sin(\omega_1-\omega)}{\sqrt{1+\tan^2\vartheta_1\sin^2\omega_1-\cos^2(\omega_1-\omega)}}\right),$$

i. e.

$$\begin{aligned}\bar{\sigma}&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\int_0^{\omega_1}\frac{\sin(\omega_1-\omega)\,d\omega}{\sqrt{1+\tan^2\vartheta_1\sin^2\omega_1-\cos^2(\omega_1-\omega)}}\\&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left[\tan^{-1}\left(\frac{\cos(\omega_1-\omega)}{\sqrt{1+\tan^2\vartheta_1\sin^2\omega_1-\cos^2(\omega_1-\omega)}}\right)\right]_0^{\omega_1}\\&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left[\tan^{-1}(\cot\vartheta_1\operatorname{cosec}\omega_1)-\tan^{-1}(\cot\vartheta_1\cot\omega_1)\right]\\&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left(\frac{\bar{\alpha}}{2ik^{\mathrm{V}}}+\frac{\bar{\beta}}{2ik^{\mathrm{V}}}-\frac{\partial}{2}\right)\\&=\mp\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left(\frac{\bar{\alpha}+\bar{\beta}+\bar{\gamma}}{2ik^{\mathrm{V}}}-\partial\right)\end{aligned}$$

Now any spherical oblique triangle whose angles are $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ can be divided into two spherical right triangles the sum of whose angles is $\bar{\alpha}+\bar{\beta}+\bar{\gamma}+2ik^{\mathrm{V}}\partial$, and the sum of the areas of these right triangles, as determined by the formula just written, *i. e.* the area of the oblique triangle, is

$$(63)\qquad\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)\left(\frac{\bar{\alpha}+\bar{\beta}+\bar{\gamma}}{2ik^{\mathrm{V}}}-\partial\right),$$

i. e. the area of any spherical triangle on a sphere of radius $\bar{r}$ *is* $\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)$ *times the excess of the sum of its angles divided by* $2ik^{\mathrm{V}}$ *over* ∂. For the plane $\bar{r}=ik\partial$, and the area is

$$(64)\qquad\frac{1}{K_2}\left(\frac{\bar{\alpha}+\bar{\beta}+\bar{\gamma}}{2ik^{\mathrm{V}}}-\partial\right).$$

This expression shows that the Euclidean area of any triangle on a sphere of radius unity bears the constant ratio K_2 to the non-Euclidean area of the plane triangle having the same angles (when $k^{\mathrm{V}}=-\frac{1}{2}i$), which is a proof, different from that given by Professor Klein (*Math. Ann.*, Vol. IV, p. 619), that the plane non-Euclidean geometry is identical with the Euclidean geometry on a surface of constant curvature K_2; and this proof proceeds directly from Gauss' definition of curvature.

These formulæ can evidently be extended to a spherical polygon of any number of sides; the area of a spherical polygon of n sides is $\frac{1}{K_2}\sin^2\left(\frac{\bar{r}}{2ik}\right)$ times the excess of the sum of its angles divided by $2ik^{\mathrm{V}}$ over $(n-2)\partial$.

The area of a conic may be found in a similar manner, namely, taking the plane of the conic as the plane $z=0$, and any two sides of that triangle which is self-conjugate with respect to the conic and also with respect to the section of the absolute by the plane $z=0$ as the axes of x and y, $\Omega = x^2+y^2-1$ and the equation of the conic is

$$\frac{x^2}{p^2}+\frac{y^2}{q^2}-1=0,$$

where $(\pm p,\ 0)$ and $(0,\ \pm q)$ are the intersections of the curve with the axes of x and y. If now $\pm\bar{a}$ and $\pm\bar{b}$ are the distances of these intersections from the origin, by (3),

$$\cos^2\left(\frac{\bar{a}}{2ik}\right)=\frac{1}{1-p^2},\ i.\,e.\ p=\pm i\tan\left(\frac{\bar{a}}{2ik}\right),$$
$$\cos^2\left(\frac{\bar{b}}{2ik}\right)=\frac{1}{1-q^2},\ i.\,e.\ q=\pm i\tan\left(\frac{\bar{b}}{2ik}\right),$$

and the equation of the conic is

$$\frac{x^2}{\tan^2\left(\frac{\bar{a}}{2ik}\right)}+\frac{y^2}{\tan^2\left(\frac{\bar{b}}{2ik}\right)}+1=0,$$

or the coordinates of any point of it may be written

$$x=p\cos\psi=i\tan\left(\frac{\bar{a}}{2ik}\right)\cos\psi,\quad y=q\sin\psi=i\tan\left(\frac{\bar{b}}{2ik}\right)\sin\psi,$$

and the area of the conic is, by (39) (putting $z=0$, $dx_1=dx$, $dy_1=0$, $dx_2=0$, $dy_2=dy$)

$$\frac{1}{K_2}\int\!\!\int\frac{dx\,dy}{(1-x^2-y^2)^{\frac{3}{2}}}=\frac{4}{K_2}\int_0^p\left[\frac{y}{(1-x^2)\sqrt{1-x^2-y^2}}\right]_{y=0}^{y=\frac{q}{p}\sqrt{p^2-x^2}}dx$$

$$=\frac{4}{K_2}\sin\left(\frac{\bar{a}}{2ik}\right)\cos^2\left(\frac{\bar{a}}{2ik}\right)\tan\left(\frac{\bar{b}}{2ik}\right)\int_0^{\frac{1}{2}\varpi}\frac{\sin^2\psi\,.\,d\psi}{\left[1-\sin^2\left(\frac{\bar{a}}{2ik}\right)\sin^2\psi\right]\sqrt{1-\left(1-\dfrac{\cos^2\left(\frac{\bar{a}}{2ik}\right)}{\cos^2\left(\frac{\bar{b}}{2ik}\right)}\right)\sin^2\psi}}$$

$$=\frac{4i}{K_2}\int_0^{F_1}\frac{\varkappa^2\operatorname{sn}a\operatorname{cn}a\operatorname{dn}a\operatorname{sn}^2u\,du}{1-\varkappa^2\operatorname{sn}^2a\operatorname{sn}^2u}=\frac{4i}{K_2}\Pi_1(u,\ a),$$

where am u, modulus $\varkappa$, and parameter a are defined thus:

$$\operatorname{am}u=\psi,\quad \varkappa'=\sqrt{1-\varkappa^2}=\frac{\cos\left(\frac{\bar{a}}{2ik}\right)}{\cos\left(\frac{\bar{b}}{2ik}\right)},\quad -\varkappa^2\operatorname{sn}^2a=-\sin^2\left(\frac{\bar{a}}{2ik}\right),$$

F_1 is the complete elliptic integral of the first kind, and $\Pi_1(u,\ a)$ is the complete elliptic integral of the third kind in Jacobi's notation.

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